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Production, illustrations, photo research, and composition by
Publication Services, Inc.

GetMath Educational Software http://www.getmath.com

ISBN 1-932661-68-9

Printed in the United States of America by Bernard J. Klein Publishing

10 9 8 7 6 5 4

8.66% APR is based on a Prime Rate of 8.25% as of August 1, 2006.

To Thomas Stewart Wesner

Dad

To Margot

Phil

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Preface

Intent

This book is designed to prepare college students for the mathematics they need in the social sciences, computer science, business and economics, and physical sciences up to the precalculus level. It is also intended to serve a course that has as its objective an introduction to or review of what are currently called "precalculus" topics. In addition, some of those topics that are amplified in modern discrete mathematics and finite mathematics courses are introduced.

Assumptions

It is assumed that students have completed an intermediate algebra course, and therefore have been introduced to solving equations, factoring, radicals and graphing linear equations. It is also assumed students own a scientific calculator and are familiar with the basic keys for arithmetic computation. Keystrokes for a typical scientific calculator are presented where they go beyond the basic arithmetic operations.

Graphing calculators/computers

The text acknowledges the growing availability of graphing calculators. These are not yet so available or even accepted that we can assume their use throughout the text. Thus, the material, as presented, does not *require* these devices, and the directions for the examples and problem sets are suitable for situations with or without a graphing device.

The text specifically indicates how to use these devices in places where graphing calculators would be appropriate. Explicit references to graphing calculators are indicated by the graphic symbol which marks this section of the preface. Examples are presented based on the TEXAS INSTRUMENTS TI-81 graphing calculator. The Computer-Aided Mathematics section introduces the basic principles involved in using a graphing calculator, employing the TI-81 as an example.

Certain exercises require extensive (nongraphing) calculator use. These are noted with the symbol .

Several exercises invite the student to write a program for a programmable calculator or computer. These are indicated by the symbol that marks this paragraph. In addition, manuals on using graphing, programmable calculators, which are specific to this text, are available from the publisher. The manuals are calculator specific and available for the most prevalent calculators.

Content

Chapter 1, Fundamentals of Algebra, should be largely review. It is the experience of most teachers that this material needs to be explicitly covered, but at a rapid pace. Students who are not prepared for this treatment are in the wrong course, and will not finish the course in any case. It is better for such students to find out early and change to a more appropriate course rather than struggle to delay the inevitable.

Chapter 2, Equations and Inequalities, also has a large percentage of review content. The instructor should note that completing the square is not introduced here, and the quadratic formula is therefore not derived at this point. The formula is adequately justified, even from a theoretical viewpoint, as an exercise. Quadratic equations are not solved by completing the square, consistent with actual practice. Completing the square is delayed until chapter 3 and coverage of the equations of circles, where it is really necessary for the first time. The part of section 2–4 that treats nonlinear inequalities, as well as section 2–5 on equations and inequalities involving absolute value, will be new for many students.

Chapter 3, Relations, Functions, and Analytic Geometry, has several objectives. The first is to introduce functions. The second is to acquaint students with

the graphs of straight lines, second-degree equations, and certain "standard" relations. The third objective is to introduce the use of linear translations and symmetry to graph appropriate relations. The fourth objective is to present the idea that analytic geometry and plane Euclidean geometry are parallel concepts, with the former modeling the latter.

Functions are introduced as sets of ordered pairs, not as an abstract function machine or rule of correspondence. The transition from functions as sets of ordered pairs to an algebraic viewpoint is smooth and natural. The viewpoint of functions as sets of ordered pairs expedites teaching (and understanding) some of the harder concepts of functions, such as the one-to-one property and inverse functions, in both this and future chapters.

Chapter 4, Polynomial and Rational Functions, and the Algebra of Functions, introduces the more important features of polynomial and rational functions as well as the "algebra" of functions. The interrelationship between zeros of a function, factors of a function, and x-intercepts of a graph is stressed. The section on the decomposition of rational functions (partial fractions) could be skipped entirely without impact on the following chapters.

Chapter 5, The Trigonometric Functions, begins with the trigonometry of right triangles and degree measure. This has the advantage of introducing substantive applications and building on things students probably already know. Analytic trigonometry, applied to any angle in standard position, is introduced next. This chapter also includes an early introduction to trigonometric equations, both identity and conditional.

Chapter 6, Radian Measure, Properties of the Trigonometric and Inverse Trigonometric Functions, first introduces radian measure for angles and arc length. Graphs and properties (i.e., periodicity) of the trigonometric functions are treated next. Period and phase shift are not treated separately but are integrated into one step. It is our experience that the graphing of quite complicated functions is easily and quickly taught and learned by the method presented here. The chapter finishes with the inverse trigonometric functions. Those types of applications necessary for the calculus are stressed.

Chapter 7, Trigonometric Equations, is a full treatment of both conditional equations and identities. It builds on the experience from chapter 5.

Chapter 8, Additional Topics in Trigonometry, covers the solution of oblique triangles using the laws of sines and cosines. It then treats vectors, complex numbers in polar form, and polar coordinates.

Chapter 9, Exponential and Logarithmic Functions, is a modern treatment of these functions. That is, only a passing reference is made to tables and to applications in calculation. Instead, the use of these functions to describe and model phenomena in the sciences and business world is stressed.

Chapter 10, Systems of Linear Equations and Inequalities, introduces these topics. The concept of elimination for solving linear systems is stressed. This is the method of choice in all applied areas. Solution by substitution is not covered in this chapter; it is covered in section 3–2 (linear systems) and chapter 11 (nonlinear systems of equations). The fundamentals of matrices and determinants, matrix algebra, and Cramer's rule are also covered; these are also important for more advanced work in this area and in courses that many students will take in other disciplines. Solving systems of equations via matrix equations is covered. There are now calculators that perform matrix arithmetic (the TI-81 is used to illustrate), and thus solving systems of equations in this way has become more important than in the past. Linear programming is introduced in this chapter for what it is, an important justification for studying systems of linear inequalities, and a topic that would be referred to in business and management courses.

Chapter 11, The Conic Sections, provides a solid introduction to the conic sections and includes a treatment of nonlinear systems of equations and the substitution method of solution of systems of equations.

Chapter 12, Topics in Discrete Mathematics, begins with the important topics of sequences and series. Section 12–3, after introducing the binomial theorem, provides more experience with manipulation of sigma notation. This is important in future courses in mathematics and economics and is quite important for computer science majors, but may be omitted without impact on future sections. Finite induction follows. In addition to higher mathematics, this topic is used in computer science in the analysis of algorithms. An introduction to combinatorics and probability is next. The final section is directed at computer science majors, and includes material that has become popular only in the last decade.

Appendixes

Appendix A is the development of several lengthy formulas, which may or may not be used by the instructor. Its inclusion within the text does not promote reading the text by the student or a better understanding of the material. Appendix B gives the answers to odd-numbered exercise problems and to all chapter review and chapter test problems. Solutions are provided for trial exercise and skill and review problems. Appendix C provides graphical material that students may want to reproduce and have handy as an aid in studying.

Features

Clarity

- Exposition—An attempt has been made to write the exposition of the material in clear, logically sequenced, understandable prose. Where the exposition builds on material from an earlier section, that section is referenced.
- Structure—The structure of each section of a chapter is designed to provide both easy reading and clear examples of the skills that students are expected to master in that section. The mix of prose and examples is designed to flow smoothly and make explicit those skills that are most important.
- Mastery points—At the end of each section is a list, called *Mastery points,* of skills from that section that students are expected to master. The testing package to support this text is based on the mastery points.

Problem sets

- Problem solutions—The answers to all problems, except even-numbered exercise problems, appear at the end of the text. The complete solutions to selected problems, indicated by boxing the problem number, also appear at the end of the text.
- Core problems—Students with a strong background in a particular topic may profit by choosing to work just those problems whose numbers are in color. In those parts of the exercises where there are sequences of similar problems, these are a subset of the problems that are sufficient to exercise the skills required for that group of problems.
- Review problems—Each problem set ends with skill and review problems. These either review old material or prepare for the next section. The solutions to these problems appear in appendix B.
- Progressive difficulty—The exercises progress from straightforward application of the material covered in the exposition to problem solving via more difficult application problems and then to problems that require some ingenuity and creativity. Those problems requiring exceptional ingenuity or amount of work are marked with the symbol . Students who confront the complete range of problems will have a good introduction to the way in which this material is bent, shaped, and modified to serve a variety of disciplines.

Review

The skill and review problems review the primary skills from previous sections and chapters. These problem sets reinforce recognition of the type of problem and appropriate solution procedures, not to provide drill of skills. These problem sets are kept short

so that their presence is an aid to progress, not an impediment. Every student is most concerned with the current material, and a long review set will not be used. The solutions to these problems also appear at the end of the text.

Closure, the last link

Each chapter terminates with a *chapter summary,* which represents the highlights of that chapter, a *chapter review,* which presents review problems from that chapter, keyed to sections, and a *chapter test,* designed to help the student practice the material as it might appear on a test, out of the context of each section. The chapter test may well be longer than what would be confronted in class—its objective is to provide material out of the context of being surrounded by similar problems. In the exercise sets, there are inevitably many clues to the method of solution, including nearby problems and temporal and physical proximity to explanations. The chapter test is an aid to make that last link in learning—recognition of problem type, with attending method of solution. The answers to all chapter review and chapter test problems appear at the end of the text.

Applications

As the title of the text indicates we have tried to provide a cross section of applications, chiefly in the problem sets. These are not intended to be obtrusive on the coverage of the mathematics, but to motivate students with a variety of interests and goals by showing that this material is used in many disciplines.

Certain applications, such as mixture problems, are considered integral to a mathematics course of this type. These are fully treated. Note that applications problems that are beyond the typical syllabus can always be done without mastering the problem domain (physics, business, finance, etc.) of the application.

Mathematics in culture

We have attempted to provide an historical aspect to the material by frequent but unobtrusive references to relevant names and dates, as well as an occasional problem taken from ancient and non-Western cultures. It is hoped that students will come to appreciate that mathematics has a historical and cultural side as well as its ''applied'' side.

Mathematics ability is not in the genes

Many other industrialized cultures understand better than ours that *anyone* can do mathematics. Americans tend to feel that difficulties in learning mathematics indicate lack of ability and that mathematics should therefore be avoided by an individual encountering problems. Other cultures react by expecting the individual to work harder, and anyone who reads the newspaper has seen reports that other cultures are correct—anyone can do mathematics.

We tend to apply this misunderstanding mostly, although not exclusively, to gender. The authors have tried to be totally nonsupportive of this misconception by using a ''gender-free'' exposition throughout the text, including problem sets. Except where discussing a specific individual, the gender-oriented pronouns are scrupulously avoided.

We have also tried to avoid cultural bias. For example we do not assume students understand simple interest, perimeter and area, or even the makeup of a standard deck of playing cards.

The electronic age is here

It is assumed throughout the text that the student always has access to a scientific or business calculator, with, as a minimum, the trigonometric and logarithmic functions. No special treatment is given to calculators–the National Council of Teachers of Mathematics' (NCTM) *Standards* advises that they be a part of every student's tool kit from the fifth grade on. We do acknowledge the modern orientation toward viewing numbers in decimal form by frequently providing the decimal approximation of answers where appropriate, such as for intercepts of graphs and answers to applications.

Pedagogical highlights—to the teacher

We want this to be a text that works. It does this if a student, with the correct level of preparation, who is willing to work, masters the material at a reasonable

level of competence, without stress caused by the text itself. We have attempted to remove those things that a text can do to introduce unwarranted stress. Some identifiable characteristics for doing this follow.

Exposition

Most students do not rely much on the exposition of the material in a textbook. This is the single largest duty of the instructor. However, when a student is motivated, or has missed a class, or has not understood the instructor, the student will hopefully turn to the text. We have been careful to make the exposition clear and logical, without making undue assumptions about prior knowledge. We have worked hard to provide a logical flow of the material. When a skill previously covered is revisited, a reference to that part of the text is presented.

The predominant goal of a course at this level is skill building. We have been careful to explicitly present a sequence of steps that applies to a particular class of problem whenever possible. These are found labeled and in boxes throughout the text. Please do not confuse this material with ''spoon feeding'' students. Most students profit from explicitly seeing the steps they are performing. To leave these steps out is to teach by example, leaving the student to deduce the steps being performed. The fact that some students, and most mathematics teachers, can learn this way does not justify teaching in this manner.

Examples

For students, this is the heart of the text. In this text the examples are carefully designed and graded to correspond well to the skills being covered and to the problem sets. The examples often provide asides that indicate what is being done at certain steps.

Problem sets

Perhaps the largest source of discouragement in a mathematics course is attempting to work a homework assignment without success. We have designed the problem sets to avoid this. Problems that emphasize skills are similar to the examples, and they are carefully graded. The solutions to a representative subset of the problems are included in appendix B. These problems are indicated in the problem sets by having the problem number in a box.

In addition, *all* of the problems in this text were solved by the authors themselves. This is the best way to ensure that the problem sets are useful and commensurate with the exposition of the material in the text. Also, many of the chapters were explicitly classroom tested in prepublication form, and all of the chapters have profited from the experience we have gained in writing previous textbooks.

Success on tests

There are several levels of understanding that must be present for students to be successful on a mathematics test, and therefore to succeed in a mathematics course. This text is designed to support students and teachers in achieving this understanding. The lowest level of understanding is pure skill. Given a certain class of problems, can students apply the procedures that solve these problems. This level of understanding is arrived at by drill. As in any good textbook the exercises in the problem sets are predominantly of this nature. These types of problems parallel specific examples in the chapter. Students require examples to learn skills. It is the responsibility of the text to provide them, in a clear, explicit manner.

The next level of understanding required is the ability to apply skills to problems where the sequence of procedures used is not clearly defined. The later problems in the exercise sets are of this nature.

The next level, which is insufficiently recognized as essential, is the ability to look at a problem, out of the context of the text, and determine which set of procedures should be applied. The student who attempts to ''solve'' the expression $x^2 - 4$, even given the instructions that often accompany a problem, has not been able to classify this problem as a factoring problem and not an equation-solving problem. This level of understanding is supported here in two ways. First, the chapter review problems are temporally separate from the coverage of the material. These problems are keyed by section, however, and there are usually several exercises that apply to each problem

classification. Thus, there are still some clues as to methodology that the student will not have on a test. The chapter test is designed to provide problems in a context containing as few clues as encountered on an exam. *This makes the chapter test an integral part of the learning process*—it is where the student obtains the ability to classify problems by type, and to recall the appropriate procedures for solving.

Critical thinking

The final level of understanding (in the implied taxonomy begun under "Success on Tests") is critical thinking or problem solving. The NCTM *Standards* ask for more of this in our mathematics education system. The implications for a mathematics course at this level are still fuzzy. Problem solving presupposes a minimal level of skill, and the primary function of this text is skill building. It would be misleading to say that this text presents problem solving in the sense of presenting "real-world" applications in which the problem domain must first be mastered. This type of material requires considerable time to cover in any meaningful way, and we have never met a mathematics teacher who felt that there was enough time in a typical curriculum to even sufficiently cover basic skills. Also, this type of material can be very disconcerting for students who do not already have a knowledge of the problem domain in the application: physics, finance, chemistry, etc.

The problem sets do contain problems, toward the end of the problem set, that involve problem solving and critical thinking, but in the context of material covered in this text. Some of these problems require real synthesis of what has been covered, and we have tried to make these problems interesting but achievable.

We believe that if a student can work, or even seriously engage, these problems, that student will be able to apply the skills learned in this text to other problem domains—in a physics, finance, or biology course—where the required subject background is obtained.

Maintaining interest

Most students feel that mathematics is a plot designed to make education painful. (If you don't believe this, ask them.) To feel good about mathematics, students need to know that they will use it in whatever career path they choose.

We have tried to show, unobtrusively, that mathematics is used across all disciplines, both by explicit statements in the text and by problems selected from many disciplines. Also we have tried to show that there is a long history to mathematics, and that it has been important across time and across cultures.

We have also tried to not stifle interest! This comes from poor exposition, unpleasant surprises in homework sets (i.e., inappropriate problems), and culturally inappropriate material (assumptions about background, sexist writing). The features of this text (above) discusses these issues.

Graphing calculators

The increased use of graphing calculators is a positive development for mathematics. It can free the student from the necessity of "plotting" points—always a tiresome and error prone process. Graphing calculators also provide new ways to visualize solutions to certain classes of problems. This text welcomes these devices for those who have access to them. The teacher will see that the text can be easily used to allow the free use of graphing calculators. Instructions in problem sets demand that such important things as asymptotes and intercepts be stated. This requires an active participation by students in completing a graph whose basic shape has been determined by a pattern in silicon crystals, while removing some drudgery.

We have tried to create a happy medium with regard to this new technology. We think we can throw out the bath water and keep the baby.

Bibliography

The following books and articles have been particularly inspirational in writing this text. This is in addition to the books and articles cited within the text itself.

The books cited here would interest students wanting to see more of the social and historical side of mathematics.

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- *Great Moments in Mathematics after 1650,* Howard Eves, The Mathematical Association of America, Washington, D.C., 1983.
- *The History of Mathematics—an Introduction,* 2nd ed., David Burton, Wm. C. Brown Publishers, 1991.
- *The Mathematics of Plato's Academy,* D. H. Fowler, Clarendon Press, Oxford, 1990.
- Several anecdotes are taken from "The Lighter Side," edited by M. J. Thibodeaux, from various issues of *The Two-Year College Mathematics Journal,* 1979 through 1980.

Supplements

For the instructor

The *Instructor's Manual* includes an introduction to the text, a guide to the supplements that accompany *College Algebra and Trigonometry,* and reproducible chapter tests. Also included are a complete listing of all mastery points and suggested course schedules based on the mastery points. The final section of the *Instructor's Manual* contains answers to the reproducible materials.

The *Instructor's Solutions Manual* contains completely worked-out solutions to all of the exercises in the textbook.

Selected *Overhead Transparencies* are available to enhance classroom presentations.

WCB Computerized Testing Program provides you with an easy-to-use computerized testing and grade management program. No programming experience is required to generate tests randomly, by objective, by section, or by selecting specific test items. In addition, test items can be edited and new test items can be added. Also included with the *WCB Computerized Testing Program* is an on-line testing option which allows students to take tests on the computer. Tests can then be graded and the scores forwarded to the grade-keeping portion of the program.

The *Test Item File* is a printed version of the computerized testing program that allows you to examine all of the prepared test items and choose test items based on chapter, section, or objective. The objectives are taken directly from *College Algebra and Trigonometry.*

For the student

The *Student's Solutions Manual* introduces the student to the textbook and includes solutions to every-other odd-numbered section exercise and odd-numbered end-of-chapter exercise problems. It is available for student purchase.

Videotapes covering the major topics in each chapter are available. Each concept is introduced with a real-world problem and is followed by careful explanation and worked-out examples using computer-generated graphics. These videos can be used in the math lab for remediation or even the classroom to motivate or enhance the lecture. The videotapes are available free to qualified adopters.

The concepts and skills developed in *College Algebra and Trigonometry* are reinforced through the interactive *Software.*

The Plotter is software for graphing and analyzing functions. This software simulates a graphing calculator on a PC. You may use it to do the technology exercises even if you don't have a graphics calculator. A manual is included that describes operations and includes student exercises. The software is menu driven and has an easy-to-use window-type interface. The high-quality graphics can also be used for classroom presentation and demonstration. Students who go on to calculus classes will want to keep the software for future use.

Acknowledgments

The authors wish to acknowledge the many reviewers of this text, both in its initial form and again after their many constructive suggestions and criticisms had been addressed. In particular, we wish to acknowledge Ruth Mikkelson, University of Wisconsin–Stout; Anna Silverstein, New York Institute of Technology; Joseph Parker, University of South Carolina and Coastal Carolina College; and Patricia A. Clark, Indiana State University. We need to acknowledge Ruth Mikkelson who did a great deal of fine work to help make the text error free. Our thanks to Linda J. Murphy, Carol Hay, and Nancy K. Nickerson of Northern Essex Community College for carefully and conscientiously checking the accuracy of the entire typeset text. We also wish to acknowledge Patricia Steele who did an outstanding job as copy editor, and who often went beyond what was required and gave excellent editorial suggestions.

Throughout the development, writing, and production of this text, two WCB employees have been of such great value they deserve special recognition: Theresa Grutz and Eugenia M. Collins.

Computer-Aided Mathematics

Introduction

The increasingly widespread availability of electronic computing and graphing devices is causing the mathematics community to rethink every aspect of mathematics education. The computer age has led to two developments that have a contradictory character. First, mathematics is used more than ever in all areas of human knowledge, and is therefore more important than ever. Second, computers can do a lot of the mathematics that formerly had to be done by hand.

Unfortunately, as the technology develops, different people have access to different levels of computing power. This has put many mathematics teachers in a quandary about what to teach, what to stress, and what, in terms of technology, to allow.

In this book we are taking a middle road. We think that graphing calculators should be used whenever they are available. We present the material in a calculator-independent fashion, for those who do not yet have access to graphing calculators, but we also show where and how a graphing calculator can be used.

The graphing calculator

As of this writing, there are a half-dozen graphing calculators on the market. The proliferation of new models and features is guaranteed. We show many examples based on the Texas Instruments TI-81 graphing calculator throughout the text. It is popular, easy to use, and similar to other brands in its use. We must assume the student will learn the specifics about a calculator from its manual.

In addition to numeric calculations, all graphing calculators have a few graphing capabilities that we use extensively:

- setting the ''range'' for the screen,
- graphing an equation in which y is described in terms of x,
- tracing and zooming, and
- finding an x- or y-intercept.

We describe the first three capabilities in this introduction, and illustrate how they are accomplished with a TI-81 graphing calculator. Finding an x- or y-intercept is discussed in section 3–1.

All students at this level have had at least an introduction to the x-y coordinate system. This topic is more formally developed in section 3–1—here we present the bare essentials very informally, by example, which is enough to describe using the graphing calculator in the first few chapters.

Set the range for the screen

We graph using the x-y rectangular coordinate system. Recall that an **ordered pair** is a pair of numbers listed in parentheses, separated by a comma. In the ordered pair (x,y) x is called the *first component* and y is called the *second component;* $(5,-3)$, $(9,3)$, $(4,\frac{2}{3})$ are examples of ordered pairs. The graphing system we use is formed by sets of vertical and horizontal lines; one vertical line is called the y-axis, and one horizontal line is called the x-axis. The geometric plane (flat surface) that contains this system of lines is called the *coordinate plane.* See figure 1.

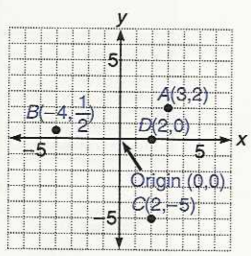


Figure 1

The **graph** of an ordered pair is the geometric point in the coordinate plane located by moving left or right, as appropriate, according to the first component of the ordered pair, and vertically a number of units corresponding to the second component of the ordered pair.

The graphs of the points $A(3,2)$, $B(-4,\frac{1}{2})$, $C(2,-5)$, and $D(2,0)$ are shown in the figure. The first and second elements of the ordered pair associated with a geometric point in the coordinate plane are called its **coordinates.**

Graphing calculators have a way to describe which part of the coordinate plane will be displayed. It is called setting the RANGE. Using the $\boxed{\text{RANGE}}$ key shows a display similar to that in table 1. The Xmin and Xmax values refer to the range of x values that will be displayed. The Ymin and Ymax values refer to the range of y values that will be displayed. The Xscl and Yscl values refer to the tick marks that will appear on the screen. The Xres refers to the number of x values that will be calculated. It should be left at 1. Throughout the text we show the Xmin, Xmax, Ymin, and Ymax values, in this order, in a box labeled RANGE. For the values shown above we would write $\boxed{\text{RANGE } -10, 10, -10, 10}$. Unless otherwise stated we assume Xscl = Yscl = 1.

Range
Xmin = −10
Xmax = 10
Xscl = 1
Ymin = −10
Ymax = 10
Yscl = 1
Xres = 1

Table 1

By entering numeric values and using the $\boxed{\text{ENTER}}$ key to move down the list, the values in the RANGE can be changed. Note that to obtain a negative number the $\boxed{(-)}$ (change sign) key is used, not the $\boxed{-}$ (subtract) key.

Figure 2 shows the screen appearance for various settings of Xmax, Xmin, Ymax, Ymin. Xscl and Yscl are 1 except where labeled Yscl=3 and Xscl=2. After setting these values with the $\boxed{\text{RANGE}}$ key, use the $\boxed{\text{GRAPH}}$ key to show the screen. Using the $\boxed{\text{CLEAR}}$ button readies the calculator for numeric calculations again. The settings in part (a) of figure 2 are the "standard" settings, obtained by selecting $\boxed{\text{ZOOM}}$ 6.

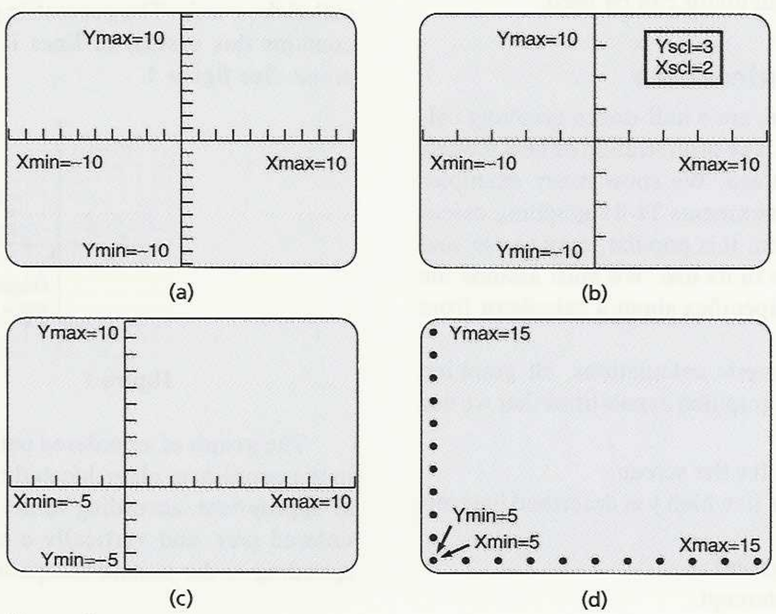


Figure 2

Observe that the distance between units is not the same on the screen. The calculator automatically makes horizontal units 1.5 times as long as vertical units. To have horizontal and vertical distances the same, use the [ZOOM] function, where option 5 says SQUARE. This makes the screen use the same scale for distance vertically and horizontally by changing the values of Xmin and Xmax.

Graph an equation in which *y* is described in terms of *x*

If an equation describes values for a variable y in terms of a variable x, the graphing calculator can be used to view the graph of the equation.

Example A

Graph each equation.

1. Graph $y = 2x - 3$.

This could be done without a graphing calculator with practically no knowledge of graphing by a table of values, by letting x take on many values, such as $-3, -2, -1, 0, 1, 2, 3$, etc., and computing y for each one. In fact, this table is shown here. The y values are computed by computing $2x - 3$ for the given x value. Each pair of values for x and y represents an ordered pair (x,y) (we always write the x value first). If we plot enough of these values in a coordinate system we start to see a picture emerge. In this case it is a straight line.

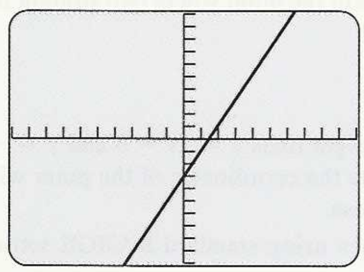
x	$y\ (2x - 3)$
-3	-9
-2	-7
-1	-5
0	-3
1	-1
2	1
3	3

Of course, the point of this section is to have the calculator automatically calculate the x and y values and plot them. Assuming the described standard RANGE settings, proceed as follows to obtain the graph:

[Y=]	Allows us to enter up to four equations
[2]	
[X|T]	The variable x
[−]	
[3]	The display looks like
[GRAPH]	

:Y_1=2X−3
:Y_2=
:Y_3=
:Y_4=

Note If there are any equations already entered for Y_1 use the [CLEAR] key before entering the equation. If there are any extra equations entered for Y_2, Y_3, or Y_4, move down with the down arrow key, ▼ to that equation and use the [CLEAR] key to clear that entry. The figure shows what the display will look like.



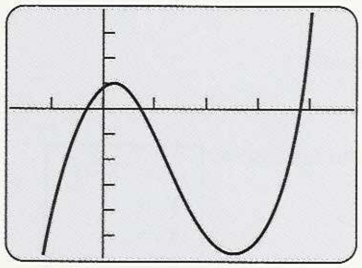
2. Graph $y = x^3 - 4x^2 + x + 1$.

The following steps would produce a graph similar to that shown in the figure.

Steps	**Explanation**
Enter the x and y-axis limits.	
[RANGE]	
[(−)] 2 [ENTER]	Xmin becomes -2.
5 [ENTER]	Xmax becomes 5.
1 [ENTER]	Xscl becomes 1.
[(−)] 6 [ENTER]	Ymin becomes -6.
4 [ENTER]	Ymax becomes 4.
1 [ENTER]	Yscl becomes 1.

Enter the function into Y_1.

[Y=] [X|T] [MATH] 3 [−] 4 [X|T] [x^2] [+]
[X|T] [+] 1 [GRAPH]



[RANGE −2,5,−6,4]

Tracing and zooming

A calculator's trace capability displays the x- and y-coordinates for a particular point. A zoom capability allows us to expand a graph around some particular point. We illustrate this here to find approximate values for the coordinates of the point where two straight lines cross.

Example B

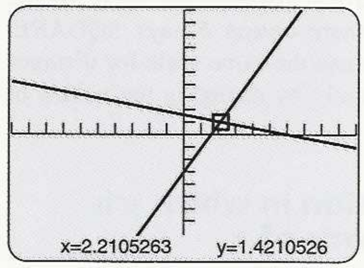
Graph the two straight lines $y = 2x - 3$ and $y = -\frac{1}{3}x + 2$. Then estimate the coordinates of the point where these two lines cross.

To graph both lines using standard RANGE settings, proceed as follows:

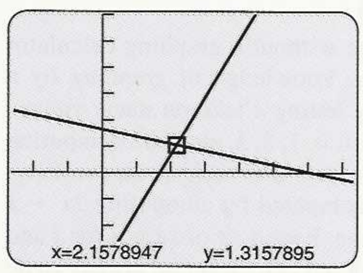
[Y =] [CLEAR] 2 [X|T] [−] 3
[ENTER] [CLEAR]
[(] [(−)] 1 [÷] 3 [)] [X|T] [+] 2
[ZOOM] 6 Standard settings

The graph shown appears.

Using the trace feature we can position the box around the point of intersection of these two lines: select [TRACE] and then use the ◀ and ▶ keys to move the blinking box as close to the point of intersection as possible. The display shows x is about 2.2 and y is about 1.4.



Using [ZOOM] 2 [ENTER] (zoom in) expands the graph around the point selected using the trace feature. It produces a new graph. Tracing shows that the coordinate of the point where the lines cross is about (2.16,1.32).



Repeatedly zooming and tracing will show that $x \approx 2.14$ and $y \approx 1.28$. Using methods shown in section 3–2 we could show that x is *exactly* $2\frac{1}{7}$ and y is exactly $1\frac{2}{7}$.

At this point you should have some idea about how to

- set the RANGE for a graph,
- graph an equation in which y is expressed in terms of x, and
- use trace and zoom to expand a particular part of a graph.

These capabilities and others are shown throughout the text, wherever they are appropriate.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C H A P T E R

1 Fundamentals of Algebra

This chapter reviews the basics of algebra. Much more can be said about everything in this chapter, but we have focused on what is essential for the rest of this book and for the mathematics most students encounter in other college courses.

1–1 Basic properties of the real number system

Suppose the postage (in cents) for a certain category of mail is as follows:

Maximum weight (in ounces)	2	3	5	10
Postage (in cents)	15	20	30	40

Above 10 ounces, the rate is 3.5 cents per ounce.

If we wanted to write a computer program to compute the postage for a given item we would have to be able to describe this situation in a mathematical way. A way of doing this is presented in this section.

The set of real numbers

The terminology of sets is often used to describe the *real number system. This is the number system we will use most of the time in this text.*

Set
A set is a collection of things. These things are called elements of the set.

The **natural numbers** is the set $N = \{1, 2, 3, \ldots\}$, which is read "N is the set of numbers one, two, three, etc." Sets are often indicated symbolically using braces "{" and "}". The three dots . . . , called an **ellipsis,** represents the repetition of a pattern.

We use the symbols[1] ε, which is read "is an element of," and $\not\varepsilon$, which is read "is not an element of." Thus for example $3 \varepsilon N$ (3 is an element of N) but $0 \not\varepsilon N$ (zero is not an element of N).

The set of **whole numbers** is $W = \{0, 1, 2, 3, \ldots\}$ and the set of **integers** is $J = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$. Every element of W is also an element of J. We say that W is a **subset** of J.

We specified the sets above by listing some of their elements, but to be able to express certain sets of numbers we need the concept of a *variable* and of *set-builder notation.* Generally a **variable** is represented by a lowercase letter, such as x or y. *A variable is a symbol that represents an unspecified element of a set that contains two or more elements.* This set is called the **replacement set.** For example,

$$\{x \mid x \varepsilon N \text{ and } x < 6\}$$

is verbalized "the set of all elements x such that x is a natural number less than six." (Remember, $<$ means "less than.") This is equivalent to the set $\{1, 2, 3, 4, 5\}$. Here x is a variable since it may represent any one of the values 1, 2, 3, 4, 5.

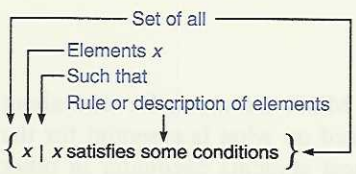


Figure 1–1

In general **set-builder notation** has the pattern shown in figure 1–1.

■ Example 1–1 A

Describe each set by listing the elements of the set.

1. $\{x \mid x \varepsilon W \text{ and } x > 3\}$
   $= \{4, 5, 6, \ldots\}$ The first element in W greater than 3 is 4

2. $\left\{\dfrac{a}{a+1} \,\middle|\, a \varepsilon \{2, 5, 8\}\right\}$ Replace a in $\dfrac{a}{a+1}$ by 2, 5, 8

   $= \left\{\dfrac{2}{2+1}, \dfrac{5}{5+1}, \dfrac{8}{8+1}\right\} = \left\{\dfrac{2}{3}, \dfrac{5}{6}, \dfrac{8}{9}\right\}$ ■

The set of **rational numbers** is $Q = \left\{\dfrac{p}{q} \,\middle|\, p, q \varepsilon J, q \neq 0\right\}$. That is, the rational numbers are written as quotients, $\dfrac{p}{q}$, of integers, where the denominator, q, may not be zero.

Every rational number has a decimal form, found by dividing the denominator into the numerator (most conveniently with a calculator). *It can be proven that the decimal form of any rational number either terminates or repeats.*

■ Example 1–1 B

Write the decimal form of each rational number.

1. $\frac{5}{8} = 0.625$ Terminating decimal
2. $\frac{2}{7} = 0.285714\overline{285714}$ Repeating decimal ■

[1] ε is the Greek letter epsilon.

Thus we could also say that

$$Q = \{x \mid \text{The decimal representation of } x \text{ terminates or repeats}\}$$

This way of describing the rational numbers gives us the tool to define the irrational numbers. The **irrational numbers** are denoted by H, where

$$H = \{x \mid \text{The decimal representation of } x \text{ does not terminate or repeat}\}$$

It can be shown that an irrational number cannot be expressed as a quotient of two integers (i.e., a rational number); in fact ''irrational'' means ''not rational.''

Examples of irrational numbers are $\sqrt{5}$ (square root of 5), $-\sqrt[3]{100}$ (opposite of the cube root of 100), and π (pi).[2] The decimal values of these numbers can be approximated using a scientific calculator as shown.

Irrational number	Calculator approximation	Typical calculator keystrokes
$\sqrt{5}$	2.236067977	5 $\boxed{\sqrt{x}}$
		TI 81: $\boxed{\text{2nd}}$ $\boxed{x^2}$ 5 $\boxed{\text{ENTER}}$
$-\sqrt[3]{100}$	−4.641588833	100 $\boxed{\sqrt[x]{y}}$ 3 $\boxed{=}$ $\boxed{+/-}$
		TI 81: $\boxed{(-)}$ $\boxed{\text{MATH}}$ 4 100 $\boxed{\text{ENTER}}$
π	3.141592654	$\boxed{\pi}$
		TI 81: $\boxed{\text{2nd}}$ $\boxed{\wedge}$

The set of **real numbers** is $R = \{x \mid x \varepsilon Q \quad \textbf{or} \quad x \varepsilon H\}$. That is, the real numbers are all numbers that can be represented by repeating, terminating, or nonrepeating and nonterminating decimals.

Note In this text, whenever a specific replacement set for a variable is not indicated it is understood that the replacement set is R.

The real number line

The **number line** (figure 1–2) helps visualize the set of real numbers. We assume that for any point on a line there is a real number, and that for every real number there is a point on the line. The number associated with a point is called the **coordinate** of that point, and the point is called the **graph** of the number. Numbers to the right of 0 are called **positive,** while those to the left are **negative.** The graph of the value zero is called the **origin.** Figure 1–2 shows the graphs for the coordinate values $-2\frac{1}{3}$, $-\sqrt{2}$, 2, 2.5, and π.

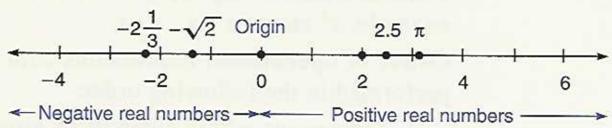


Figure 1–2

[2]The use of the Greek letter pi to represent the number it represents was introduced in 1706 in a book by the Englishman William Jones.

Properties of the real numbers

There are certain rules, called axioms, that we assume all real numbers obey. In the following axioms we assume that there are two operations called addition and multiplication. The variables a, b, and c represent any real numbers; $a + b$ represents their sum, a real number; and ab or $a \cdot b$ represents their product, a real number.[3]

Axiom	For addition	For multiplication
Commutative	$a + b = b + a$	$ab = ba$
Associative	$a + (b + c) = (a + b) + c$	$a(bc) = (ab)c$
Identity	There is a unique number 0 such that $a + 0 = a$.	There is a unique number 1 such that $a(1) = a$.
Inverse	For every a there is a value $-a$ such that $a + (-a) = 0$.	For every a except 0 there is a value $\frac{1}{a}$ such that $a\left(\frac{1}{a}\right) = 1$.
Distributive	$a(b + c) = ab + ac$	

Note 0 is called the additive identity, 1 is the multiplicative identity, $-a$ is the additive inverse of a, and $\frac{1}{a}$ is the multiplicative inverse of a.

We make the following definitions using the properties above.

Subtraction: $a - b = a + (-b)$

Division: $a \div b = a\left(\frac{1}{b}\right)$

Factor: In the product ab, a and b are called factors.

Term: In the sum $a + b$, a and b are called terms.

Like terms: Terms with identical variable factors. Like terms can be combined by combining the numerical factors. For example, $2a + 3a = 5a$, and $-3xy^2 + xy^2 = -2xy^2$.

Note $3xy$ and $3xy^2$ are not like terms and cannot be combined into one term using addition or subtraction.

Expression: A meaningful collection of variables, constants, grouping symbols, and symbols of operation. For example, $x + 3(5 - y)$ is an expression.

Natural number exponents: If $n \in N$ then a^n means n factors of a. For example, x^4 means $x \cdot x \cdot x \cdot x$.

Order of operations: Expressions with mixed operations should be performed in the following order:

1. Operations within symbols of grouping (parentheses, fraction bars, radical symbols)
2. Indicated powers and roots

[3]The juxtaposition of symbols, as in xy, to indicate multiplication is due to René Descartes (1637).

3. Multiplications and divisions from left to right
4. Additions and subtractions from left to right

Fraction: A fraction is an expression of the form $\frac{a}{b}$, $b \neq 0$.

Operations for fractions: (assuming $a, b, c, d \in R$ and $c, d \neq 0$)

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c} \qquad \frac{a}{c} + \frac{b}{d} = \frac{ad+bc}{cd}$$

$$\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c} \qquad \frac{a}{c} - \frac{b}{d} = \frac{ad-bc}{cd}$$

$$\frac{a}{c} \cdot \frac{b}{d} = \frac{ab}{cd} \qquad \frac{a}{c} \div \frac{b}{d} = \frac{a}{c} \cdot \frac{d}{b},\ b \neq 0$$

$$\frac{a}{c} \pm \frac{b}{d} = \frac{ad \pm bc}{cd}$$

Figure 1–3

The patterns for the addition and subtraction of fractions with unlike denominators can be viewed as three multiplications, indicated in figure 1–3 by arrows (1, 2, and 3) in the order in which the products are formed. The symbol ± is read "plus or minus."

■ ***Example 1–1 C***

Perform the indicated operations.

1. $5x^2 \cdot 2x^3$
   $= 10(x \cdot x)(x \cdot x \cdot x) = 10x^5$

2. $-4x^2y + 6x^2y - 2xy$
   $= 2x^2y - 2xy$

3. $\frac{2x}{y} \cdot \frac{3x}{4y} = \frac{\cancel{2}^1 x}{y} \cdot \frac{3x}{\cancel{4}^2 y}$ Divide out common factors
   $= \frac{3x^2}{2y^2}$

4. $\frac{2x}{y} + \frac{3x}{4} = \frac{(2x)(4) + (y)(3x)}{4y}$ Use $\frac{2x}{y} \times \frac{3x}{4}$
   $= \frac{8x + 3xy}{4y}$

■

An expression like -3^2 often causes confusion. It is *not* the same as the expression $(-3)^2$.

$$-3^2 \quad \text{means} \quad -(3^2) \quad \text{which is } -(3 \cdot 3) = -9$$

That is, *square the value 3 first,* then take the opposite of the result.

$$(-3)^2 \quad \text{means} \quad (-3)(-3) = 9.$$

That is, *change the sign of 3 first,* then square the result. For example,

$$15 - 3^2 = 15 - 9 = 6.$$

On a calculator,

-3^2 is calculated as 3 $\boxed{x^2}$ $\boxed{+/-}$

$(-3)^2$ is calculated as 3 $\boxed{+/-}$ $\boxed{x^2}$

Order

The set of real numbers is ordered. That is, for any two distinct numbers, one is greater than the other. If value a is greater than value b we write $a > b$. The symbol $>$ is read "**is greater than.**" Similarly, $a < b$ means that the value a **is less than** the value b. These symbols of inequality are called **strict inequalities.** The symbols $\geq$ and $\leq$ are read "**is greater than or equal to**" and "**is less than or equal to.**" These are called **weak inequalities.**[4]

The fact that the real numbers are ordered is called the law of trichotomy.

> **Law of trichotomy**
>
> For any real numbers a and b, exactly one of the following is true:
>
> $a > b, \qquad a = b, \qquad a < b$

We can determine which of the three possibilities is true in one of two ways. The easiest is to observe that $a > b$ if and only if a is to the right of b on the number line. The second method is with the following formal definition of inequality.

> **Definition of $a > b$**
>
> If $a - b$ is a positive value, then $a > b$.

Intervals

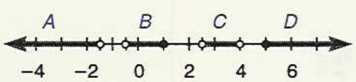


Figure 1–4

Most of the time we simply rely on our mental picture of the number line to determine which of two values is greater. We also use the concept of the number line, as well as the terminology of set-builder notation, to talk about **intervals** on the number line. Consider figure 1–4. The figure shows four intervals, A, B, C, and D. A is an interval that has no lower limit; the open circle shows that it also does not include the point at $-1\frac{1}{2}$. In B the solid circle is used to emphasize that the graph of 1 is included in the interval. C represents all values between 2.5 and 4, but specifically excludes 2.5 and 4 themselves. D is an interval with no upper limit. These intervals can be described using either set-builder notation or **interval notation.** These descriptions are shown in figure 1–5. Observe in the figure that the symbols "(" and ")" are associated with strict inequalities, and that "[" and "]" are associated with weak inequalities. We also use the symbol ∞ (infinity) to indicate the concept of no upper or lower limit. We use the symbols "$(-\infty$" to indicate an interval with no lower limit, and the symbols "$+\infty)$" to indicate that an interval has no upper limit.

The notation $2\frac{1}{2} < x < 4$ (interval C in figure 1–5) is read "x is greater than $2\frac{1}{2}$ *and* x is less than 4." This notation is called a **compound inequality.**

[4]The symbols $>$ and $<$ are attributed to Thomas Harriot in his *Artis analyticae praxis,* 1631. The symbols $\geq$ and $\leq$ were used by the Frenchman P. Bougher in 1734.

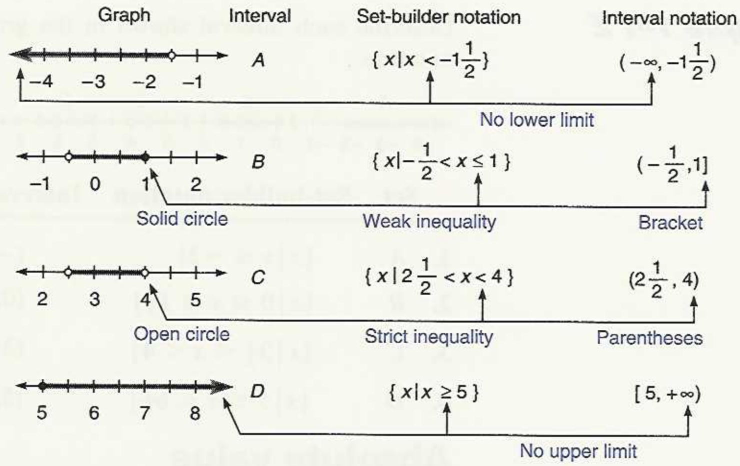


Figure 1–5

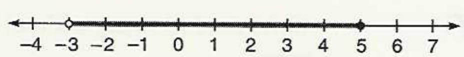
> **Compound inequality: $a < x < b$**
> $a < x < b$ means that x is greater than a and x is less than b.
>
> ***Concept***
> x is between a and b.

A similar statement applies (as in B in figure 1–5) when the symbol $\leq$ is used. Compound inequalities correspond to intervals on the number line.

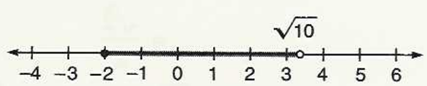
■ *Example 1–1 D*

Graph the interval, then describe the interval in set-builder notation if given in interval notation, and describe in interval notation if given in set-builder notation.

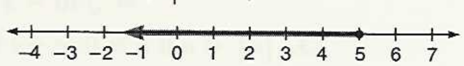
1. $\{x \mid -3 < x \leq 5\}$
Interval notation: $(-3, 5]$

Graph: 

2. $[-2, \sqrt{10})$ $\sqrt{10} \approx 3.2$ (calculator)
Set-builder notation: $\{x \mid -2 \leq x < \sqrt{10}\}$

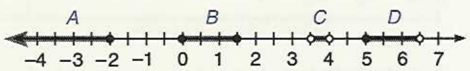
Graph: 

3. $(-\infty, 5]$
Set-builder notation: $\{x \mid x \leq 5\}$

Graph:  ■

■ ***Example 1–1 E***

Describe each interval shown in the graph in both set-builder and interval notation.



	Set	Set-builder notation	Interval notation
1.	A	$\{x \mid x \le -2\}$	$(-\infty, -2]$
2.	B	$\{x \mid 0 \le x \le 1\frac{1}{2}\}$	$[0, 1\frac{1}{2}]$
3.	C	$\{x \mid 3\frac{1}{2} < x < 4\}$	$(3\frac{1}{2}, 4)$
4.	D	$\{x \mid 5 \le x < 6\frac{1}{2}\}$	$[5, 6\frac{1}{2})$

Absolute value

The absolute value of a real number measures the undirected distance that the number is from the origin. Note that an **undirected distance** is nonnegative (positive or zero). The symbol $|x|$ is read "the absolute value of x."[5] The formal definition of absolute value uses the fact that the opposite of a negative number is positive.

Absolute value

$$|x| = \begin{cases} x \text{ if } x \text{ is positive or zero} \\ -x \text{ if } x \text{ is negative} \end{cases}$$

Concept

If a number is nonnegative, its absolute value is the number itself. If a number is negative, the absolute value of that number is its opposite.

Note The symbol $-x$ does not necessarily represent a negative number. It symbolically states the opposite of the value that x represents.

■ ***Example 1–1 F***

Write the following without absolute value by using the definition of absolute value.

1. $\left| -\dfrac{\sqrt{2}}{2} \right| = -\left(-\dfrac{\sqrt{2}}{2}\right) = \dfrac{\sqrt{2}}{2}$ $\qquad -\dfrac{\sqrt{2}}{2} < 0;\ |a| = -a \text{ if } a < 0$

2. $|3 - \sqrt{10}| = -(3 - \sqrt{10}) = -3 + \sqrt{10} = \sqrt{10} - 3$ $\qquad 3 - \sqrt{10} < 0$

3. $|a| = a$ if $a \ge 0$ or $-a$ if $a < 0$.

[5]The symbol $|x|$ was introduced by the German mathematician Karl Weierstrass in 1841.

Properties of absolute value

There are certain properties that can be proved valid concerning absolute value for any real numbers a, b. The proof would rely on the previously stated axioms for the real numbers as well as the definition of absolute value stated above. These properties are:

[1] $|a| \geq 0$ [2] $|-a| = |a|$

[3] $|a| \cdot |b| = |ab|$ [4] $\dfrac{|a|}{|b|} = \left|\dfrac{a}{b}\right|$

[5] $|a - b| = |b - a|$

■ Example 1–1 G

Use the preceding properties and the definition of absolute value to simplify and remove the absolute value symbol from the following expressions.

1. $|2a|$

$= \|2\| \cdot \|a\|$	$\|a\| \cdot \|b\| = \|ab\|$
$= 2\|a\|$	$\|2\| = 2$
$= 2a$ if $a \geq 0$ or $-2a$ if $a < 0$	Definition of $\|a\|$

2. $|x^2y|$

$= \|x^2\| \cdot \|y\|$	$\|a\| \cdot \|b\| = \|ab\|$
$= x^2\|y\|$	$\|x^2\| = x^2$ since $x^2 \geq 0$
$= x^2y$ if $y \geq 0$ or $-x^2y$ if $y < 0$	Definition of $\|y\|$

Mastery points

Can you

- List the elements of a set when the set is described in set-builder notation?
- Find the decimal form of any rational number, and describe it as terminating or repeating?
- Combine simple algebraic expressions?
- Recognize and describe intervals in set-builder, interval, and graph form?
- Find the absolute value of expressions?
- Use properties of absolute value to simplify expressions?

Exercise 1–1

Describe each set by listing the elements of the set.

1. $\{x \mid 3 < x < 12 \text{ and } x \in W\}$
2. $\{x \mid x \in W \text{ and } x \notin N\}$
3. $\{x \mid x \text{ is odd and } x \in N \text{ and } x < 21\}$
4. $\left\{x \mid x \text{ is represented by a digit in the decimal expansion of } \dfrac{178}{185}\right\}$
5. $\{3x \mid x \in \{-2, -1, 0, 1, 2, 3, 4\}\}$
6. $\left\{\dfrac{3x}{x+1} \,\middle|\, x \in \{1, 2, 3, 4, 5\}\right\}$

Give the decimal form of each rational number. Describe the form as terminating or repeating, as appropriate.

7. $\frac{2}{5}$ **8.** $\frac{2}{3}$ **9.** $\frac{3}{13}$ **10.** $\frac{1}{18}$

Simplify the given algebraic expressions.

11. $-5(2-8)^2 - 12(11-3)$

12. $(-8^2-3) - 5(2-5)^2 + (1-4)(4-1)$

13. $\frac{5}{8} - \frac{3}{5} + \frac{3}{4}$

14. $\frac{1}{2} - \frac{1}{4} + \frac{2}{3}$

15. $\frac{8}{15} \div \left(\frac{5}{9} - \frac{3}{10}\right)$

16. $\left(\frac{3}{7} - \frac{7}{12}\right) \div \left(\frac{3}{7} + \frac{7}{12}\right)$

17. $\dfrac{-3[5-2(9-12)+4]-(8-2)(2-8)}{5(3-7)^2-(3-7)^3}$

18. $\dfrac{-5(8)-(-2)(4)^2}{3^2-5^2} + \dfrac{18}{6-4(-4)}$

19. $\dfrac{x}{a} - \dfrac{y}{b}$

20. $\dfrac{2a-3}{3a} + \dfrac{3b+2}{2b}$

21. $\dfrac{x-y}{4x} - \dfrac{2x+y}{3y}$

22. $\dfrac{4a}{b} - \dfrac{2a}{3c} + \dfrac{1}{2}$

23. $(7x^2)(-2x^5)$

24. $(-3xy)(2xy^2z)$

25. $\dfrac{3x}{2y} \div \dfrac{5y}{2x} \cdot \dfrac{x}{5y}$

26. $\dfrac{5a}{3b} \cdot \dfrac{a}{2b} \div \dfrac{2b}{a}$

27. $\left(\dfrac{a}{2b} + \dfrac{b}{a}\right) \cdot \dfrac{3b}{5a}$

28. $\dfrac{3y^6}{4a^2} - \dfrac{2y^2}{4a^2}$

Graph the interval and describe in interval notation.

29. $\{x \mid -2 < x < 8\}$

30. $\{x \mid 0 < x \le 10\}$

31. $\{x \mid -8 \le x < 0\}$

32. $\{z \mid -3\frac{1}{4} \le z < -2\frac{3}{4}\}$

33. $\{y \mid -\sqrt{2} < y \le \pi\}$

34. $\{x \mid -\frac{1}{3} < x < \frac{2}{3}\}$

35. $\{x \mid x < 4\}$

36. $\{y \mid y \le -1\}$

37. $\{x \mid x \ge -2\}$

Graph the interval and describe in set-builder notation.

38. $[-2, 5]$

39. $[-5, -1]$

40. $[-\frac{2}{3}, -\frac{1}{3})$

41. $(-\infty, 1]$

42. $(1.8, +\infty)$

43. $\left[-\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$

Describe each interval shown in the graph in both set-builder and interval notation.

44. A **45.** B **46.** C **47.** D **48.** E **49.** F **50.** G **51.** H

Write the expression without the absolute value symbol.

52. $-\left|8\frac{1}{2}\right|$

53. $|-4|$

54. $\left|-3\frac{1}{3}\right|$

55. $-|2|$

56. $|-\sqrt{3}|$

57. $|-\sqrt{10}-3|$

58. $-|\sqrt{10}-6|$

59. $\left|2\frac{1}{4}-4\right|$

60. $|6-\sqrt{10}|$

61. $|(-5)^2|$

62. $-|-\sqrt{2}|$

63. $-|\sqrt{2}-3|$

64. $|x^6|$

65. $|2x^4|$

66. $\left|\dfrac{y^4}{3}\right|$

67. $\left|\dfrac{x^2y^6}{z^8}\right|$

Use the properties for absolute value and the definition of absolute value to simplify the following expressions and rewrite without the absolute value symbol.

68. $|3a|$

69. $-|-5x^2|$

70. $\left|\dfrac{3x^2}{2y}\right|$

71. $\left|\dfrac{5x}{2y^2}\right|$

72. $|5x-10y|$

73. $|(x-2)^2(x+1)|$

74. $\dfrac{|x|}{x}$

75. $\dfrac{x^2}{|x|}$

76. Put the following fractions in order, from smallest to largest: $\frac{3}{8}, \frac{5}{17}, \frac{21}{38}, \frac{22}{39}, \frac{15}{42}, \frac{13}{40}$. *Hint:* Use the decimal form of the number.

77. Compute the average of the following temperatures: -8, -6, 5, 2, 4. To compute the average one adds up the values and divides by the number of values.

$\boxed{78.}$ Suppose the postage (in cents) for a certain category of mail is as follows:

Maximum weight (in ounces)	2	3	5	10
Postage (in cents)	15	20	30	40

Above 10 ounces, the rate is 3.5 cents per ounce. For example, any piece of mail weighing greater than 2 ounces and less than or equal to 3 ounces will cost 20 cents to mail.

a. Graph the weights as intervals on the number line.
b. Describe the intervals using interval notation.
c. Compute the postage rate in cents per ounce for each interval, to the nearest 0.1 cents.

For example, for the first interval it is $\dfrac{15 \text{ cents}}{2 \text{ ounces}} = 7.5$ cents per ounce.

79. Some calculators have a key labeled $\boxed{a^{b}/_{c}}$. This computes fraction expressions in exact form. For example $\frac{1}{3} + \frac{1}{3}$ results in $\frac{2}{3}$, not 0.6666666. If your calculator has such a key use it to compute the result of problems 13 through 16.

Skill and review

1. Simplify $2x^2 \cdot 3x^3$.
2. Simplify $2n + (-2n)$.
3. Simplify $8 - (-3)$.
4. $2{,}000{,}000{,}000 =$ **a.** $2 \cdot 10^{10}$ **b.** $2 \cdot 10^9$ **c.** 2^{10} **d.** 2^9
5. $0.00003 =$ **a.** $\dfrac{3}{1{,}000}$ **b.** $\dfrac{3}{10{,}000}$ **c.** $\dfrac{3}{100{,}000}$ **d.** $\dfrac{3}{1{,}000{,}000}$
6. Calculate $-3[2(4[\frac{1}{2}(2 - 3) + 2] - 1) + 7] + 4$.
7. Multiply $2a(2a - 2b - ac)$.
8. Multiply $(2a - c)(3a + 2c)$.

1–2 Integer exponents and polynomials

The amount of solar radiation reaching the surface of the earth is about 3.9 million exajoules a year. (An exajoule is one billion joules of energy.) The combustion of a ton of oil releases about 45.5 joules of energy. How many tons of oil would have to be burned to equal the total amount of solar radiation reaching the surface of the earth in 1 hour? The annual consumption of global energy is about 350 exajoules. How many tons of oil would have to be burned to yield this amount of energy?

Questions like this are very important to those trying to find safe, alternate, renewable fuels for society. The concept of exponents is very helpful in dealing with this kind of problem, and exponents is one of the subjects of this section.

Integer exponents

We defined natural number exponents in section 1–1. For example, x^4 means $x \cdot x \cdot x \cdot x$. It has proved useful to extend the definition of exponents to include a definition for an exponent of zero and a definition for negative exponents.[6]

[6]It was understood by 1553, by Michael Stifel, that a quantity with an exponent zero had the value one. The superscript notation for positive exponents was introduced by Descartes in 1637. Newton introduced our modern notation for negative (and fractional) exponents in a letter in 1676. He wrote "Since algebraists write a^2, a^3, a^4, etc. for *aa, aaa, aaaa,* etc., so I write . . . a^{-1}, a^{-2}, a^{-3}, etc. for $\dfrac{1}{a}, \dfrac{1}{aa}, \dfrac{1}{aaa}$, etc."

Zero exponent

$$a^0 = 1 \text{ if } a \neq 0$$

Negative integer exponents

If $n \in N$, then

$$a^{-n} = \frac{1}{a^n} \text{ if } a \neq 0$$

■ ***Example 1–2 A***

Simplify each expression; assume no variable expression represents zero.

1. $5^0ab^0 = (1)a(1) = a$

2. $(a - 3b)^0 = 1$

3. $5^{-2} = \frac{1}{5^2} = \frac{1}{25}$

4. $(x + y)^{-1} = \frac{1}{x + y}$ ■

There are several properties that apply to expressions with integer exponents.

Properties of integer exponents

If $a, b \in R$ and $m, n \in J$, then

[1] $a^m a^n = a^{m+n}$

[2] $\frac{a^m}{a^n} = a^{m-n}, a \neq 0$

[3] $(ab)^m = a^m b^m$

[4] $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$

[5] $(a^m)^n = a^{mn}$

■ ***Example 1–2 B***

Simplify each expression. Leave the answer with only positive exponents. Assume no variable represents zero.

1. $(5x^3y^4)(-2x^2y^2) = -10x^{3+2}y^{4+2}$ $\quad a^m a^n = a^{m+n}$

$= -10x^5y^6$

2. $\frac{12x^8y^5}{3x^3y^{-1}} = \frac{12}{3}x^{8-3}y^{5-(-1)}$ $\quad \frac{a^m}{a^n} = a^{m-n}$

$= 4x^5y^6$

3. $(a^3)^4(b^4)^{-1} = a^{12}b^{-4}$ $\quad (a^m)^n = a^{mn}$

$= \frac{a^{12}}{b^4}$

4. $\left(\frac{3x^2}{y^5}\right)^3 = \frac{(3x^2)^3}{(y^5)^3}$ $\quad \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

$= \frac{27x^6}{y^{15}}$ $\quad (ab)^m = a^m b^m$

5. $2x^{-2}y + x^2y^{-1} = \frac{2y}{x^2} + \frac{x^2}{y}$

$= \frac{2y^2 + x^4}{x^2y}$

6. $\dfrac{x^n y^{2m}}{x^2 y^m} = x^{(n-2)}y^{(2m-m)}$ $\qquad \dfrac{a^m}{a^n} = a^{m-n}$

$\qquad = x^{n-2}y^m$

Scientific notation

Very large and very small numbers appear in most of the physical and social sciences. For example, the mass of a hydrogen atom is 0.000 000 000 000 000 000 000 001 67 gram, and there are over 5,000,000,000 (5 billion) people on this planet. The English language probably permits at least 8,000,000,000,000 three-word sentences.

If we observe that 1,000,000 (one million) is 10^6, and $\dfrac{1}{1{,}000{,}000}$ (one-millionth) is 10^{-6}, we see that integer exponents might prove useful in expressing these quantities.

We can convert any decimal number into what is called **scientific notation.** We define this form of a number Y to be

$$Y = a \times 10^n$$

where $1 \leq |a| < 10$. The steps to put Y into this form are as follows.

> **To put a number *Y* into scientific notation**
>
> 1. Move the decimal point to the position immediately following the first nonzero digit in Y.
> 2. The absolute value of n is the number of decimal places that the decimal point was moved.
> 3. If the decimal point is moved *left, n* is *positive.*
>   If the decimal point is moved *right, n* is *negative.*
>   If the decimal point is not moved, n is 0.

To put a number back into decimal form we reverse these steps.

To understand why these steps are correct consider the value 53,000:

$$\begin{aligned} 53{,}000 &= 53{,}000 \times 1 \\ &= 53{,}000 \times (10^{-4} \times 10^4) \\ &= (53{,}000 \times 10^{-4}) \times 10^4 \\ &= 5.3 \times 10^4 \end{aligned}$$

■ ***Example 1–2 C***

Convert each number into scientific notation.

1. $3{,}500{,}000\,000 = 3.500\,000\,000 \times 10^9 = 3.5 \times 10^9$

2. $-0.000\,000\,000\,000\,805\,4 = -0.000\,000\,000\,000\,8.054 \times 10^{-13}$

$\qquad = -8.054 \times 10^{-13}$

■ ***Example 1–2 D***

Convert each number into a decimal number.

1. -2.8903×10^{12}
$= -2{,}890{,}300{,}000{,}000$ Move decimal 12 places to the right

2. 2.8903×10^{-10}
$= 0.000\ 000\ 000\ 289\ 03$ Move decimal 10 places to the left ■

Polynomials

Terms

$$5x^4y - 2x^3y^3 + 4x^2y^3 - 12$$

Factors of term

Recall from section 1–1 that a **term** is part of a sum and a **factor** is a part of a product. The constant factor of a term is called its **numerical coefficient** (-2 in $-2x^3y^2$). An **algebraic expression** is any meaningful collection of variables, constants, grouping symbols, and symbols of operations. A special kind of expression is a polynomial.

> **Polynomial**
> A **polynomial** is an expression containing one or more terms in which each factor is either a constant or a variable with a natural number exponent.

A **monomial** is a polynomial of one term, a **binomial** is a polynomial of two terms, and a **trinomial** is a polynomial of three terms.[7]

A polynomial is an expression in which the variables are not found in radicals or denominators, and have only natural number exponents. The idea is that a polynomial is made up of constants and variables using only the operations of addition, subtraction, and multiplication—as a result, a polynomial is always defined. Observe that $\frac{1}{x}$ uses division. Thus it is not a polynomial, and is not defined for the replacement value zero.

■ ***Example 1–2 E***

The following illustrate monomials, binomials, and trinomials.

1. Monomials: 5, $3x^2$, $2x^3y^5$, $-\sqrt{2}x$, $\frac{1}{2}x$ (note that $\sqrt{2}$ and $\frac{1}{2}$ are constants)

2. Binomials: $5x^2 + 2$, $-4a + 3b$, $(3x^4 - 2y^5) - z^4$

3. Trinomials: $2x^5 - 3x^2 + 5$, $3x^2 - x + 12$, $(a + 2b)^2 - 3(a + 2b) + 7$ ■

The following expressions are not polynomials: $5\sqrt{x}$, $\frac{5}{x}$, and $5x^{-3}$. They include operations other than addition, subtraction, and multiplication.

[7]Mono, bi, tri, and poly are prefixes taken from the Greek language. They mean "one," "two," "three," and "many," respectively.

We often categorize polynomials in one variable by their degree.

Degree
The **degree of a term** of a polynomial in one variable is the exponent of the variable factor. The degree of a constant term is zero. The **degree of a polynomial** is the degree of its term of highest degree.

For example, the degree of $4x^5$ is 5, the degree of $2x^5 - 3x^2$ is 5, and the degree of $3x^4 + 2x^3 - 11x^2 - 5$ is 4.

Substitution of value

It is important to remember that an expression represents a numerical value. Something like $a + b$ makes sense only if a and b represent actual real numbers. To find the value of an expression, given a value for each variable in an expression, use what we will call **substitution of value.** This means to *replace each variable in the expression by the value associated with that value.* Then perform the indicated arithmetic.

■ Example 1–2 F

Find the value that each expression represents when $a = -5$, $b = 4$, $c = -\frac{1}{2}$, and $d = 6$.

Problem	Solution
1. $5a^2b$	$5(-5)^2(4) = 5(25)(4) = 500$
2. $(2a - \frac{1}{2}b)(6c + d)$	$[2(-5) - \frac{1}{2}(4)][6(-\frac{1}{2}) + 6]$
	$= [-10 - 2][-3 + 6] = -36$ ■

Polynomials are a very important part of mathematics. There is an "arithmetic" of polynomials. We will review how to add, subtract, multiply, and divide polynomials.

Multiplication of polynomials

Multiplication of monomials proceeds using the properties for exponents covered earlier in this section. Basically, *multiply coefficients and add exponents.* For example,

$$(4x^2y)(-3x^5y^3) = -12x^7y^4$$

Multiplication of general expressions uses the distributive axiom. Recall that this states that

$$a(b + c) = ab + ac$$

■ Example 1–2 G

Multiply.

1. $-3a^2b(5a^2 - 3ab)$

$$= -3a^2b(5a^2 - 3ab)$$
$$= -3a^2b(5a^2) + 3a^2b(3ab) = -15a^4b + 9a^3b^2$$

2. $(5a - 3b)(2x + 3y)$
In this case we apply the distributive axiom to multiply each term of $(5a - 3b)$ by $(2x + 3y)$. This amounts to multiplying each term in the second factor, $2x$ and $3y$, by each term in the first factor, $5a$ and $-3b$. This is illustrated as

Multiply by $5a$

$$(5a - 3b)(2x + 3y) = (5a - 3b)(2x + 3y)$$

Multiply by $-3b$

$$= 5a(2x + 3y) - 3b(2x + 3y)$$
$$= 10ax + 15ay - 6bx - 9by$$

3. $(2a - 3b - 5c)(4x - 7y + z)$
In this case each term in the second factor is multiplied by each term in the first factor; this is a total of nine products.

$$(2a - 3b - 5c)\overbrace{(4x - 7y + z)}$$

$$2a\,(4x - 7y + z) - 3b\,(4x - 7y + z) - 5c\,(4x - 7y + z)$$
$$= 8ax - 14ay + 2az - 12bx + 21by - 3bz - 20cx + 35cy - 5cz$$ ■

Addition and subtraction of polynomials

Addition and subtraction is equivalent to combining like terms. Basically, *add and subtract coefficients—do not change exponents.* Recall from section 1–1 that like terms are terms with identical variable factors. The distributive axiom gives the method for combining like terms. For example, by the distributive axiom,

$$5a^2b + 8a^2b = (5 + 8)a^2b$$
$$= 13a^2b$$

We also need to observe that $-(a + b) = -a - b$. That is, *when a grouping symbol is preceded by a negative sign, the grouping symbol may be removed if we change the sign of every term within the grouping symbol.* This can be viewed as multiplication by -1; that is, $-(a + b) = (-1)(a + b)$, which is $-a - b$.

■ ***Example 1–2 H***

Perform the indicated operations.

1. $(5x^2 - 3x + 4) - (2x^2 - 6x + 9) + (6 - 3x - x^2)$

$= 5x^2 - 3x + 4 - 2x^2 + 6x - 9 + 6 - 3x - x^2$ — Rewrite without parentheses

$= 5x^2 - 2x^2 - x^2 - 3x + 6x - 3x + 4 - 9 + 6$ — Terms reordered for clarity

$= 2x^2 + 1$ — Combine like terms

2. $(2x - 3)(8x + 3)$

$= 16x^2 + 6x - 24x - 9$ — Multiply

$= 16x^2 - 18x - 9$ — Combine like terms ■

Division of polynomials

Division by a monomial uses the rules for exponents as well as the division of numbers: $\dfrac{6x^6}{3x^3} = 2x^3$. That is, basically, *divide coefficients and subtract exponents.* If the numerator has more than one term we *rewrite the quotient using separate fractions:* $\dfrac{8x^4 - 6x^2 + 12x^2y}{2x^2}$ should be rewritten as $\dfrac{8x^4}{2x^2} - \dfrac{6x^2}{2x^2} + \dfrac{12x^2y}{2x^2}$. Simplifying each fraction produces the result

$$4x^2 - 3 + 6y$$

To divide by a polynomial of more than one term we use the **long division algorithm.** An algorithm is a precise procedure for doing something. The long division algorithm repeats the same three steps over and over until the degree of the remainder is less than the degree of the divisor:

1. Divide the first term of the divisor into the first term of the dividend.
2. Multiply the divisor by the result.
3. Subtract and bring down remaining terms.

It is practically the same as the long division used in arithmetic. This algorithm is illustrated in parts 3 and 4 of the following example.

■ ***Example 1–2 I***

Divide.

Problem	Solution
1. $\dfrac{24x^6y^3}{36x^2y^3}$	$\dfrac{24}{36} \cdot \dfrac{x^6}{x^2} \cdot \dfrac{y^3}{y^3} = \dfrac{2}{3}x^4(1) = \frac{2}{3}x^4$
2. $\dfrac{24x^6y^3 - 20x^2y^8 + 12x^2y^2}{12x^2y^2}$	$= \dfrac{24x^6y^3}{12x^2y^2} - \dfrac{20x^2y^8}{12x^2y^2} + \dfrac{12x^2y^2}{12x^2y^2}$ $= 2x^4y - \frac{5}{3}y^6 + 1$

3. $\dfrac{4x^3 - 2x^2 - 8}{2x + 1}$.

Division by an expression of more than one term; use long division. $2x + 1$ is the divisor and $4x^3 - 2x^2 - 8$ is the dividend. We insert a term $0x$ for clarity.

$$\begin{array}{r@{}l}
 & \;2x^2 - 2x + 1 \\
2x + 1 & \overline{)\;4x^3 - 2x^2 + 0x - 8} \\
 & \underline{-(4x^3 + 2x^2)} \\
 & \qquad -4x^2 + 0x - 8 \\
 & \qquad \underline{-(-4x^2 - 2x)} \\
 & \qquad\qquad\quad 2x - 8 \\
 & \qquad\qquad\quad \underline{-(2x + 1)} \\
 & \qquad\qquad\qquad\; -9
\end{array}$$

Thus, $(4x^3 - 2x^2 - 8) \div (2x + 1) = 2x^2 - 2x + 1$ with a remainder of -9. We would usually write the result as

$$\frac{4x^3 - 2x^2 - 8}{2x + 1} = 2x^2 - 2x + 1 - \frac{9}{2x + 1}.$$

To check, we would compute $(2x + 1)(2x^2 - 2x + 1) - 9$, and make sure the result is $4x^3 - 2x^2 - 8$.

4. $\dfrac{2x^4 + x^3 - 3x^2 + 3}{x^2 - 2x + 1}$. The divisor has more than one term so use long division.

$$\begin{array}{r}
2x^2 + \phantom{x^3 - {}} 5x + 5 \\
x^2 - 2x + 1 \overline{)\; 2x^4 + x^3 - 3x^2 + 0x + 3} \\
\underline{-(2x^4 - 4x^3 + 2x^2) \phantom{{}+ 0x + 3}} \\
5x^3 - 5x^2 + 0x + 3 \\
\underline{-(5x^3 - 10x^2 + 5x) \phantom{{}+ 3}} \\
5x^2 - 5x + 3 \\
\underline{-(5x^2 - 10x + 5)} \\
5x - 2
\end{array}$$

Thus, $\dfrac{2x^4 + x^3 - 3x^2 + 3}{x^2 - 2x + 1} = 2x^2 + 5x + 5 + \dfrac{5x - 2}{x^2 - 2x + 1}$. ■

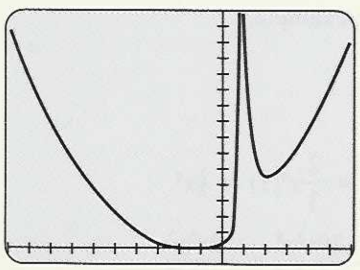


Figure 1–6

When using the long division algorithm it is important to arrange the terms of the divisor and dividend in decreasing order of degree. The algorithm stops when the degree of the remainder is less than that of the divisor.

Note A graphing calculator can actually be used to help verify that the last calculation was correct. The graph of $Y_1 = (2x^4 + x^3 - 3x^2 + 3)/(x^2 - 2x + 1)$ is shown in figure 1–6 (using x-values from -10 to 6, and y-values from -2 to 150). The graph of $Y_2 = 2x^2 + 5x + 5 + ((5x - 2)/(x^2 - 2x + 1))$ is identical to it. Thus, graphically, $Y_1 = Y_2$ for any value of x. This would indicate our result was correct.

Subscripted variables

Subscripted variables are used in many situations. For example, if one measured the temperature of something at two different times these temperatures might be called t_1 and t_2 (read "t sub 1 and t sub 2"). The 1 and 2 are subscripts, in this case indicating the first and second temperature. *Variables with different subscripts are not like terms,* and we never perform any arithmetic operations on subscripts. Thus, for example, $t_1 + 4t_1 + t_2$ can combine into $5t_1 + t_2$, but that is all. Similarly,

$$\begin{aligned}(t_2 - t_1)(2t_2 + t_1) &= 2t_2^2 + t_1t_2 - 2t_1t_2 - t_1^2 \\ &= 2t_2^2 - t_1t_2 - t_1^2\end{aligned}$$

Mastery points

Can you

- Simplify expressions involving integer exponents?
- Convert numbers between scientific notation and decimal notation?
- Determine whether an expression is a monomial, binomial, or trinomial, or not a polynomial at all?
- Evaluate expressions when given values for the variables, using substitution of value?
- Perform the indicated operations of addition, subtraction, multiplication, and division to combine and transform expressions?

Exercise 1–2

Use the properties of exponents to simplify the following expressions.

1. $2x^5 \cdot x^2 \cdot x^4$
2. $-2x^4 \cdot (2^2x^3)$
3. $(-2^2)(2^3)$
4. $(-2)^2(-3^2)$
5. $(3a^5b)(2a^2b^2)$
6. $(-4x^5yz^2)^2(xy^3z^2)$
7. $2x^{-2}(2^6x^3)$
8. $3a^{-1}b^4(3^{-1}a^2b)$
9. $3x^4y^{-3}$
10. $-5x^{-2}y$
11. $(2x^3y^5)^3$
12. $(-3a^2bc^3)^4$
13. $(-3^2a^2b^{-3})^2$
14. $(2x^{-3}y^2)^3$
15. $\dfrac{1}{x^2y^{-3}}$
16. $\dfrac{2}{2^{-1}x^{-5}}$
17. $(-3)^{-3}$
18. $\dfrac{1}{2^{-2}}$
19. $(-2x^{-2})^2$
20. $(2^{-3}x^3y)^2$
21. $(2x^4y)(-3x^3y^{-2})$
22. $(3^2x^{-5}y)(-2^2x^5y^{-4})$
23. $(6x^4)^0$
24. $\dfrac{3}{x^{-2}(y^4)^5}$
25. $\dfrac{3^2x^{-4}y}{3xy^{-4}}$
26. $\dfrac{2x^{-2}y^0z^3}{-2^{-3}x^2y^2z^{-1}}$
27. $\left(\dfrac{9a^5b^4}{18ab^{11}}\right)^{-3}$
28. $\left(\dfrac{6x^5y^2}{2x^2y^{-2}}\right)^2$
29. $\left(\dfrac{-2x^2y}{5x^{-3}y^3}\right)^3$
30. $\left(\dfrac{3x^2y^{-2}}{6x^{-3}y}\right)^0$
31. $\left(\dfrac{3^3a^{-4}bc^6}{3^4ab^{-2}c^{-2}}\right)^2$
32. $\left(\dfrac{4a^{-3}}{12a^{-5}}\right)^2$
33. $\left(\dfrac{3x^2}{12x^4}\right)\left(\dfrac{16x^{-1}}{2x}\right)$
34. $\left(\dfrac{a^2b^3c^0}{a^3b^{-4}c^2}\right)\left(\dfrac{a^4c}{a^2b}\right)$
35. $\left(\dfrac{x^2y}{x^2y^2}\right)^2\left(\dfrac{3x^0y^{-2}}{4xy}\right)^2$
36. $\left(\dfrac{a^5bc^2}{2abc}\right)^3\left(\dfrac{2^2a^3b^2c}{ab^{-2}c^3}\right)^{-1}$
37. $x^{2n-3}x^{2n+3}$
38. $a^nb^{2n}(a^2b^{-n})$
39. $\dfrac{x^{n-2}}{x^{n-3}}$
40. $\dfrac{2^{-3}a^{-n}}{2ab^{-2}}$
41. $\left(\dfrac{x^ny^{2-n}}{x^{-n}y^{n-2}}\right)^4$
42. $\left(\dfrac{x^{2n}}{x^{n+1}}\right)^{-3}$

Convert each number into scientific notation.

43. 3,650,000,000,000,000
44. 2,003,000,000,000
45. −19,002,000,000,000
46. 0.000 000 000 203
47. −0.000 000 000 000 029 2
48. −0.000 000 5
49. 0.000 000 000 003 502
50. −21,500,000,000,000

Convert each number given in scientific notation into a decimal number.

51. 2.502×10^{13}
52. 4.31×10^{-8}
53. -1.384×10^{-10}
54. -5.11×10^7
55. 9.23×10^6
56. 3.002×10^{12}

57. The mass of an electron is

0.000 000 000 000 000 000 000 000 000 91 gram.

Put this value into scientific notation.

58. The half-life of lead[8] 204 is

14,000,000,000,000,000,000 years.

Put this value into scientific notation.

Categorize each expression as a monomial, binomial, trinomial, polynomial (if more than three terms), or not a polynomial. Also state the degree of those expressions that are polynomials.

59. $5x^2 - 3x - 2$
60. $2x - 1$
61. $4x^3 - 3x^2 - x + 3$
62. $3x^5 - 2x^4 + x^2 - 3x - 7$
63. $\sqrt{3}\,x^4 - \frac{1}{2}x^6 + 4$
64. $3(x + 1) + (3x - 2) + 9$
65. $3\sqrt{x} - 4x - 2$
66. $9x^2$

In the following problems find the value that each expression represents, assuming that $a = -5$, $b = 4$, $c = -\frac{1}{2}$, $d = 6$.

67. $3a^3 - 5a^2 + 11a - 2$
68. $(a - 2b)(3c + d)$
69. $\frac{1}{3}a - 2cd^2 + \sqrt{b}$
70. $(3a^2 - 2a + 1)(a - 1)^2$
71. $2c^3 - 5c + bc$
72. $-2(4[\frac{1}{2}(-b + 3) + 2] - 1) + 7$

Perform the indicated operations.

73. $(5x^2 - 3x - 2) - (3x^2 + x - 8)$
74. $(-3x + 4x^2 + 11) + (4 - 2x - 5x^2)$
75. $(2a - 3b - c) + (5a - 4b + 2c) - a$
76. $-(a - 4b) + (2b - 3a) - (5a - 5b)$
77. $(3x^2y - 2xy^2 - xy) + (2xy^2 - 5x^2y + 3xy)$
78. $(5ab^2 - 2a^2b^2 + 3a) - (a^2b^2 - 2ab^2 + 3a^2)$
79. $3x - [x - y - (7x - 3y)]$
80. $7x - [4x + 3y + (x - 2y)]$
81. $[-(3a - b) - (2a + 3b)] - [(a - 6b) - (3b - 10a)]$
82. $-(a - 4) + [3a - (2a + 5) - 4]$
83. $2x^2(5x^3 - 2x^2 + 7)$
84. $-5xy^2(2x^2 - 3xy - y^2)$
85. $-2a^2b(5a^2b + 3a^2b^2 - 2ab^3)$
86. $\frac{1}{2}ab^4(4a^4b - 8a^3b^2 + 2a^2b^3 - 10)$
87. $(5a - 3)(5a + 3)$
88. $(2x + 4y)(x - 2y)$
89. $(3x + y)(5x - y)$
90. $(a - 3b)(2a + 7b)$
91. $(2a - b)(a - 2b + c)$
92. $(x + 3y)(2x - y - 4)$
93. $(2x^2 - 3x + 1)(5x^2 - 2x + 7)$
94. $(a^2 + a - 3)(2a^2 - 6a + 4)$
95. $(5b^2 - 2b + 3)(b^2 + b - 3)$
96. $(4x^2 - 5x - 3)(x^2 + 2x + 1)$
97. $(x + 2y)(x - 3y)(x + y)$
98. $(2a - b)(a + b)(a - b)$
99. $(3a + 2b)(3a - b)(a + 2b)$
100. $(x + 4)(2x - 3)(2x + 5)$
101. $(2a - 3)(3a - 2b + c)$
102. $(3x - 2)^2(x + 5)$
103. $(x - 2y)(x + 2y)^2$
104. $(a + 2b)(a - b)^2$
105. $(2x + 5)^3$
106. $(a - 2b)^3$

Perform the indicated divisions.

107. $\dfrac{12x^5y^2}{18x^2y}$
108. $\dfrac{24a^8b^3c}{8a^4bc}$
109. $\dfrac{6a^6b^2 - 8a^4b^4 + 12a^4b^6}{2a^4b^2}$
110. $\dfrac{20x^2y^3 + 5xy^5 - 10xy^3}{10xy^3}$
111. $\dfrac{10x^4y^2z^2 + 15x^3y^2z - 20x^2y^4}{15x^2y^2}$
112. $\dfrac{-30a^9b^3 - 12a^6b^3 + 18a^3b^3}{12a^3b^3}$
113. $\dfrac{x^2 - 3x + 2}{x - 1}$
114. $\dfrac{y^3 - 1}{y - 1}$
115. $\dfrac{x^4 - 3x^3 + 8}{x + 2}$
116. $\dfrac{2z^5 - 3z^3 + z^2 - z - 1}{z + 1}$
117. $\dfrac{6x^3 + x^2 - 10x + 5}{2x + 3}$
118. $\dfrac{2x^4 - 3x + 5}{x - 3}$
119. $\dfrac{4x^3 - x^2 + 3x + 4}{x - 2}$
120. $\dfrac{2x^2 - 11x + 3}{x - 8}$
121. $\dfrac{4x^3 - x^2 + 5}{x^2 - x + 1}$
122. $\dfrac{2x^4 - x^3 + 2x - 2}{x^2 + 3}$
123. $\dfrac{3x^4 - 2x^2 - x + 1}{x^2 - 3}$
124. $\dfrac{x^3 - x^2 + 2x - 5}{x^2 - 2x - 1}$

[8]The time necessary for half of the material present to decay radioactively.

125. Perform the indicated operations on the subscripted variables.

a. $(3t_1 + 4t_2) - (t_1 + 6t_2 - 3t_3)$

b. $(3t_1 + 4t_2)(t_1 + t_2 - 3t_3)$

c. $(-3x_1^2x_2^4)(2x_1x_2)^3$

126. Divide.

a. $(2x^3 - 3x^2 - x + 4) \div (2x + 3)$

b. $(2x^3 - 3x^2 - x + 4) \div (2x^2 + 3x - 1)$

c. $(2x^3 - 3x^2 - x + 4) \div (2x^3 + 3x^2 - 3x + 2)$

127. The equations

$$\begin{aligned}(a^2 + b^2)(c^2 + d^2) &= (ac + bd)^2 + (ad - bc)^2 \\ &= (ac - bd)^2 + (ad + bc)^2\end{aligned}$$

played important roles in medieval algebra and, still, in trigonometry, an important part of mathematics. Show that all three expressions are equivalent.

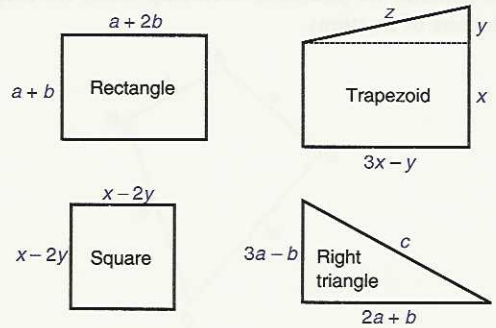
128. Srinivasa Ramanujan (1897–1920) was a mathematical genius from India at the beginning of this century. Among numerous incredible accomplishments at the highest levels of mathematics, he also developed and used the following formula:

$$\begin{aligned}&(a + 1)(b + 1)(c + 1) + (a - 1)(b - 1)(c - 1) \\ &= 2(a + b + c + abc)\end{aligned}$$

Compute the left member to show that it is the same as the right member.

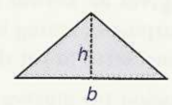
129. In a certain class, four one-hour tests are given and a final exam. The final exam counts 40% of the course grade, so each test counts 15%. Under these conditions the course grade G is described by the formula $G = 0.15(T_1 + T_2 + T_3 + T_4) + 0.4E$, where T_1 represents the grade of the first test, etc., and E is the grade on the final exam. Compute a student's final exam grade, to the nearest 0.1, if the student's test grades were 68, 78, 82, and 74, and the final exam grade was 81.

130. The perimeter of a geometric object is the distance around the object. Write and simplify an expression for the perimeter of each of the objects shown in the figure.



131. The area of a square or rectangle is the product of its two dimensions. Find an expression for the area of the square and rectangle shown in the figure with problem 130.

132. The area of a triangle is $\frac{1}{2}bh$ (see the figure), where b means base and h means height. Find and simplify an expression for the area of the right triangle in the figure with problem 130.



133. Find and simplify an expression for the area of the trapezoid shown in the figure with problem 130. *Hint:* The figure can be decomposed into a rectangle and a triangle.

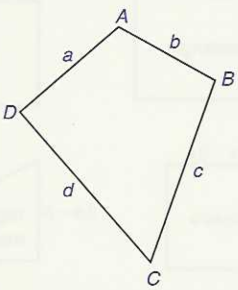
$\boxed{\textbf{134.}}$ In his book *The Schillinger Method of Musical Composition,* Schillinger shows how to apply algebraic concepts to the study of music.[9] One application is rhythmic continuity. Since $(A + B)^2 = A^2 + AB + BA + B^2$, if we let $A = 1$ and $B = 2$ we obtain $9 = 3^2 = (1 + 2)^2 = 1 + 2 + 2 + 4 = 9$, and if we let $A = 2$ and $B = 1$ we obtain $9 = 3^2 = (2 + 1)^2 = 4 + 2 + 2 + 1$. These are two sequences of attacks, which total 9 beats.

a. Use this idea to generate two sequences of attacks that total 16 beats.

b. Use the expansion of $(A + B + C)^2$ with values for A, B, and C to generate two sequences that total 36 beats.

[9]The authors acknowledge former student and professional musician, Gary Leach, for bringing this to their attention.

135. The Rhind Mathematical Papyrus is an Egyptian work on mathematics. It dates to the sixteenth century B.C., and contains material from the nineteenth century B.C. It contains 84 problems, including tables for manipulations of fractions.



Problems 51–53 of the Papyrus include the following formula for finding the area of a four-sided figure like $ABCD$ in the figure: $\frac{1}{2}(a+c)\cdot\frac{1}{2}(b+d)$. (Observe that $\frac{1}{2}(a+c)$ is the average length of the two sides a and c; the same is true for $\frac{1}{2}(b+d)$.) Show that this formula is equivalent to

$$\tfrac{1}{4}(ab + ad + bc + cd)$$

By the way, this formula is inaccurate. Except for rectangles, it gives an answer that is too large. It was used for the purpose of taxing land. There is not always an economic incentive to get the correct answer.

136. One early notation for algebra put the coefficient first and the exponent last, as we do today.[10] Exponents were not written above the line, however. Thus, a^2 was $a2$, and $5a^2b$ was $5a2b$. Using this notation, simplify the following expressions. Write the result in the early notation.

a. $(3a2b3)(2ab2)$ **b.** $(ab3)(3ab)$
c. $(aaa)(3abb)$ **d.** $(5a5)(2b2)$
e. $\dfrac{9a9}{3a3}$

137. Most computers take longer to multiply two values than to add or subtract them. Suppose a certain computer takes five units of time for a multiplication but only one for an addition/subtraction.

Now consider the equivalent expressions $a(b+c) = ab + ac$. The expression $a(b+c)$ represents one addition followed by one multiplication. This would take $1 + 5 = 6$ units of time to calculate. The expression $ab + ac$ has two multiplications followed by one addition. This would take 11 units of time. Thus, given a choice, the expression $a(b+c)$ should be used when writing programs for this computer.

Analyze each of the following expressions for this computer by performing the indicated operations and simplifying, then comparing the number of units of time each expression would take. The number of units are shown for the given expression.

a. $(a+b)(c+d) : 7$ **b.** $(2a+3b)(a-2b) : 22$
c. $a(b+c-d) : 7$ **d.** $(a+b)(c+d+e) : 8$
e. $(a+b)(c+d)(e+f) : 13$

Any scientific calculator will accept numbers in scientific notation. In fact this is the only way to enter a very large or small number. Calculators usually have a key marked EE, for enter exponent, or EXP, for exponent. For example, to perform a calculation such as

$$(3.8 \times 10^{8})(-2.5 \times 10^{-12})$$

one would enter a sequence like

3.8 $\boxed{\text{EE}}$ 8 $\boxed{\times}$ 2.5 $\boxed{+/-}$ $\boxed{\text{EE}}$ 12 $\boxed{+/-}$ $\boxed{=}$

TI-81: 3.8 $\boxed{\text{EE}}$ 8 $\boxed{\times}$ $\boxed{(-)}$ 2.5 $\boxed{\text{EE}}$ $\boxed{(-)}$ 12 $\boxed{\text{ENTER}}$

Example

Calculate (3,500,000,000,000,000)(51,000,000,000). Leave the result in scientific notation.

$$\begin{aligned}(3{,}500{,}000{,}000{,}000{,}000)&(51{,}000{,}000{,}000)\\ &= (3.5 \times 10^{15})(5.1 \times 10^{10})\\ &= (3.5)(5.1)(10^{15} \times 10^{10})\\ &= 17.85 \times 10^{25}\\ &= 1.785 \times 10^{26}\end{aligned}$$

On a calculator the steps would be 3.5 $\boxed{\text{EE}}$ 15 $\boxed{\times}$ 5.1 $\boxed{\text{EE}}$ 10 $\boxed{=}$. On the TI-81 use $\boxed{\text{ENTER}}$ for $\boxed{=}$. ■

Compute the values in problems 138–141 using a calculator.

138. (31,000,000,000,000)(5,300,000,000,000,000)
139. (5,000,000,000,000) ÷ 0.000 000 000 25
140. $(39{,}100{,}000{,}000)^3$
141. $\sqrt{4{,}000{,}000{,}000{,}000{,}000{,}000}$

[10]Pierre Hérigone, a French mathematician, advocated this in his book *Cursus mathematicus,* which appeared in 1634.

142. The amount of solar radiation reaching the surface of the earth is about 3.9 million exajoules a year. An exajoule is one billion joules of energy. The combustion of a ton of oil releases about 45.5 joules of energy.

a. Compute how many tons of oil would have to be burned to equal the total amount of solar radiation reaching the surface of the earth in one hour.

b. The annual consumption of global energy is about 350 exajoules. How many tons of oil would have to be burned to yield this amount of energy?

Skill and review

1. Write 360 as a product of prime integers. Prime integers are the positive integers greater than one that are not divisible by any other positive integer except one. They are 2, 3, 5, 7, 11, 13, 17, etc.
2. $3x^5y^4 - 12x^3y^3 + 6x^2y^3 = 3x^2y^3(\_\_\_\_ - \_\_\_\_ + \_\_\_\_)$
3. $x^2 - 16 = (x - 4)(\_\_\_\_ + \_\_\_\_)$
4. $x^2 + 6x + 8 = (x + 2)(\_\_\_\_ + \_\_\_\_)$
5. $x^2 + 6x - 16 = (x - 2)(\_\_\_\_ + \_\_\_\_)$
6. $6x^2 - 7x - 3 = (2x - 3)(\_\_\_\_ + \_\_\_\_)$
7. $(x - 1)(x^2 + x + 1) = \_\_\_\_$

1–3 Factoring

A computer program must calculate the quantity $A^2 - B^2$ for thousands of different values of A and B. Also, the computer can store only the first six digits of any number. Since $A^2 - B^2$ is the same as $(A - B)(A + B)$, would one form be better than the other to use in the program?

As a matter of fact, the form $(A - B)(A + B)$ is better. It requires only one multiplication, whereas $A^2 - B^2$ is $AA - BB$, and requires two. Multiplication is much slower than addition on many computers. (See problem 137 in section 1–2, also.) The form $(A - B)(A + B)$ is a factored form of $A^2 - B^2$.

Also, the factored form can be more accurate than the unfactored form. Suppose the computer can store only two more digits after the leftmost nonzero digit of a number[11] (in reality they can store about five more such digits, but the idea is the same in either case). Let A be 3.00 and B be 2.99. Then consider the two calculations, where all intermediate results cannot contain more than two digits after the leftmost nonzero digit.

$(A - B)(A + B)$	$A^2 - B^2$	
$(3.00 - 2.99)(3.00 + 2.99)$	$3.00^2 - 2.99^2$	
$0.0100(5.99)$	$9.00 - 8.9401$	
0.0599	$9.00 - 8.94$	Round 8.9401 to three digits
	0.0600	

Thus the first computation, using the factored form, gives the accurate value 0.0599, whereas the second gives 0.0600. This error is only one part in one thousand, but can be important in some situations.

[11] In science this is called three significant digits.

This example illustrates one of the ways in which factoring of algebraic expressions can be useful. As we will see in later sections, another important use is in simplifying algebraic fractions. In this section we will investigate several ways in which algebraic expressions can be "factored." Recall that a factor is part of an indicated product—to factor an expression means to write it as a product.

The greatest common factor

This method of factoring applies to expressions with two or more terms and *should always be tried first.* The greatest common factor (GCF) is the greatest expression that divides into each term of the expression. The variables in the GCF are those that appear in every term. The exponent on a variable in the GCF is the *smallest exponent of that variable* found in the terms.

■ Example 1–3 A

Factor.

1. $6x^5y^2 + 12x^3y^4 - 48x^5y^2z^3$ — The GCF is $6x^3y^2$

$= \underline{6x^3y^2} \cdot x^2 + \underline{6x^3y^2} \cdot 2y^2 - \underline{6x^3y^2} \cdot 8x^2z^3$

$= 6x^3y^2(x^2 + 2y^2 - 8x^2z^3)$ — This expression is a product

2. $-20x^6y^3z + 12x^2y^4 - 4x^2y^3$

When the leading coefficient is negative, we usually factor out the negative of the GCF; the GCF is $4x^2y^3$, so we will factor out $-4x^2y^3$.

$= (-4x^2y^3)(5x^4z) + (-4x^2y^3)(-3y) + (-4x^2y^3)(1)$

$= -4x^2y^3(5x^4z - 3y + 1)$

3. $3a(x - 4) + 2b(x - 4) - x + 4$

$= 3a(x - 4) + 2b(x - 4) - (x - 4)$ — Group the last two terms

$= (x - 4)(3a + 2b - 1)$ — Factor out the GCF $(x - 4)$ ■

Factoring by grouping

Some expressions with four or more terms can be factored as illustrated in the following example.

■ Example 1–3 B

Factor $2am^2 - bn^2 - bm^2 + 2an^2$.

There are no common factors in the first two or second two terms. Therefore we try a different arrangement.

$= 2am^2 - bm^2 + 2an^2 - bn^2$ — Rearrange the order of the terms

$= m^2(2a - b) + n^2(2a - b)$ — Factor the common factor from the first two and second two terms

$= (2a - b)(m^2 + n^2)$ — Factor out the common factor $(2a - b)$ ■

Quadratic trinomials

Many three-termed expressions are quadratic trinomials.

> **Quadratic trinomial**
> A quadratic trinomial is a polynomial of the form
> $$ax^2 + bx + c$$
> We will concern ourselves with the case where $a, b, c \in J$.

There are several ways to factor quadratic trinomials,[12] but we will illustrate factoring by inspection. This method relies on the fact that if a quadratic trinomial factors, it factors into a product of two binomials.

■ ***Example 1–3 C***

Factor.

1. $9x^2 - 21x - 8$

There are only two possibilities for the first term in the binomial factors, since we need to form $9x^2$ in the first term:

$$(9x + \cdots)(x + \cdots) \quad \text{or} \quad (3x + \cdots)(3x + \cdots)$$

To obtain -8 in the third term, the missing terms in our binomial are either $1, -8$ or $-1, 8$, or $2, -4$ or $-2, 4$. By trying these possibilities and then multiplying we obtain the correct factors.

$$\begin{array}{lcl} (9x + 1)(x - 8) & \text{or} & (3x + 1)(3x - 8) \\ (9x - 1)(x + 8) & \text{or} & (3x - 1)(3x + 8) \\ (9x + 8)(x - 1) & \text{or} & (3x + 8)(3x - 1) \\ (9x - 8)(x + 1) & \text{or} & (3x - 8)(3x + 1) \\ (9x + 2)(x - 4) & \text{or} & (3x + 2)(3x - 4) \\ & \text{etc.} & \end{array}$$

The product $(3x - 8)(3x + 1)$ is $9x^2 - 21x - 8$, so these are the factors.

2. $x^2 + 3x - 28$

$(x + \cdots)(x + \cdots)$, with missing factors of $1, -28$ or $-1, 28$ or $4, -7$ or $-4, 7$. Trial and error produces the result $(x - 4)(x + 7)$. ■

The difference of two squares

> **Difference of two squares**
> An expression of the form $a^2 - b^2$ is a difference of two squares.

[12]In particular the algorithmic method presented in *Intermediate Algebra with Applications* by Terry H. Wesner and Harry L. Nustad, by the same publisher as this text. This method does not depend upon trial and error.

It is easy to verify that

$$a^2 - b^2 = (a - b)(a + b)$$

Thus, a difference of two squares can always be factored into two binomials that are identical except for the sign of the second term. Binomials with this property are called **conjugates.**

Note that *the exponents of any variable that is a perfect square is even.* To see why, observe that $(x^n)^2$ describes any variable being squared, and $(x^n)^2 = x^{2n}$, and $2n$ is an even number.

Example 1–3 D

Factor the following.

1. $x^4 - 81$

$= (x^2)^2 - 9^2$ — $(x^2)^2 = x^4$, $9^2 = 81$

$= (x^2 - 9)(x^2 + 9)$

$= (x - 3)(x + 3)(x^2 + 9)$ — $x^2 - 9$ is a difference of two squares

Note *A sum of two squares,* such as $x^2 + 9$, *does not factor* using real numbers.

2. $x^6 - 4$

$= (x^3)^2 - 2^2$ — $x^6 = (x^3)^2$

$= (x^3 - 2)(x^3 + 2)$

The difference and sum of two cubes

Difference and sum of two cubes

An expression of the form $a^3 - b^3$ is a difference of two cubes.
An expression of the form $a^3 + b^3$ is a sum of two cubes.

Each of these two types of expressions can be written as the product of a binomial and trinomial in the following manner:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$
$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

The following illustrates how to remember these two patterns. It is illustrated for the difference of two cubes.

$a^3 - b^3$ = binomial · trinomial

The binomial is the difference of the two factors that were cubed: $(a - b)$

The trinomial can be formed from the binomial in the following way:

1. Square the first term of the binomial: $(a - b)(a^2$
2. Use the opposite of the sign in the binomial: $(a - b)(a^2 +$
3. Multiply the values of the two terms in the binomial (disregard the minus sign): $(a - b)(a^2 + ab$
4. Square the second term of the binomial: $(a - b)(a^2 + ab + b^2)$

The same method is valid for the sum of two cubes. See part 2 of example 1–3 E. Also, note that *to find the factor that was cubed* (i.e., a and b), *divide any exponents by 3.* It is also useful to know the cubes of the first few natural numbers:

$$1^3 = 1,\ 2^3 = 8,\ 3^3 = 27,\ 4^3 = 64,\ 5^3 = 125$$

■ Example 1–3 E

Factor the following.

1. $x^3 - 27$

Form the binomial:	$x - 3$	$x^3 - 27 = x^3 - 3^3$
Now form the trinomial:		
Square the first term:	$(x - 3)(x^2$	
Change the sign:	$(x - 3)(x^2 +$	
Multiply the two terms:	$(x - 3)(x^2 + 3x$	
Square the last term:	$(x - 3)(x^2 + 3x + 9)$	

Thus, $x^3 - 27 = (x - 3)(x^2 + 3x + 9)$.

2. $x^6 + 64y^3$

Binomial:	$(x^2 + 4y)$	$x^6 + 64y^3 = (x^2)^3 + (4y)^3$
Square the first term:	$(x^2 + 4y)(x^4$	
Change the sign:	$(x^2 + 4y)(x^4 -$	
Multiply the two terms:	$(x^2 + 4y)(x^4 - 4x^2y$	
Square the last term:	$(x^2 + 4y)(x^4 - 4x^2y + 16y^2)$	

Thus, $x^6 + 64y^3 = (x^2 + 4y)(x^4 - 4x^2y + 16y^2)$. ■

It is a good idea to review when each factoring method is used.

Factoring techniques

When factoring, remember the following:

- Whenever possible, factor out any common factor (i.e., the GCF).
- When there are two terms, think about a difference of two squares or sum/difference of two cubes.
- When there are an even number of terms, and more than two terms, think about grouping.
- When there are three terms, think about a quadratic trinomial.

Substitution for expression

The following two expressions are both quadratic trinomials of the form $y^2 - 2y - 3$; in one the variable y is replaced by $(z + 1)$, and in the other by $(2x - 1)$.

$$y^2 - 2y - 3$$
$$(z + 1)^2 - 2(z + 1) - 3$$
$$(2x - 1)^2 - 2(2x - 1) - 3$$

Since they are of the same form they factor in a similar manner. The method of **substitution for expression** can help factor expressions like the last one. To use this form of substitution, *replace any expression that appears more than once with some variable.* After factoring *replace this variable with the original expression.* This method is a form of the general concept of substitution, a very useful theme that we will see many times throughout this text. (Substitution of value, section 1–2, was another instance of this concept.)

Example 1–3 F

Factor using substitution.

1. $(x-4)^2 + x - 4 - 6$

$(x-4)^2 + (x-4) - 6$	
$z^2 + z - 6$	Replace $(x-4)$ by z
$(z-2)(z+3)$	Factor
$[(x-4)-2][(x-4)+3]$	Replace z by $(x-4)$
$(x-6)(x-1)$	Simplify each factor

2. $(2x-3)^3 - 27$

$(2x-3)^3 - 27$	
$z^3 - 27$	Replace $(2x-3)$ by z
$(z-3)(z^2+3z+9)$	Factor
$[(2x-3)-3][(2x-3)^2+3(2x-3)+9]$	Replace z by $(2x-3)$
$(2x-6)(4x^2-6x+9)$	Simplify each factor
$2(x-3)(4x^2-6x+9)$	Common factor in $(2x-6)$

Several methods of factoring must be used to factor some expressions.

Example 1–3 G

Factor.

$a^2(4-x^2) + 8a(4-x^2) + 16(4-x^2)$

$= (4-x^2)(a^2+8a+16)$	Factor out the common factor $(4-x^2)$
$= (2-x)(2+x)(a+4)^2$	$4-x^2$ is a difference of two squares and $a^2+8a+16$ is a quadratic trinomial

Mastery points

Can you

- Factor using
  1. greatest common factor?
  2. grouping?
  3. inspection (quadratic trinomials)?
  4. difference of two squares?
  5. sum and difference of two cubes?
- Factor using substitution for expression?

Exercise 1–3

Factor the following expressions using the greatest common factor.

1. $12x^2 - 9xy - 18$
2. $30a^3b^2 - 10ab + 60ab^2$
3. $-20a^4b^2 + 60a^3b - 24a^2b^2$
4. $-40x^4y^3 + 16x^3y^4 + 20x^3y^3$
5. $6x(a - b) + 5y(a - b)$
6. $2a(x + 3) - b(x + 3)$
7. $5a(2x - y) - 2x + y$
8. $3x(4a + 3b) - 4a - 3b$
9. $2m(n + 5) - n - 5 - p(n + 5)$
10. $m - 2n - 5a(m - 2n) + 2b(m - 2n)$

Factor the following expressions using grouping.

11. $ac + ad - 2bc - 2bd$
12. $2ax + 6bx - ay - 3by$
13. $5ax - 3by + 15bx - ay$
14. $3ac - 2bd - 6ad + bc$

Factor the following expressions by inspection.

15. $6x^2 + 13x + 6$
16. $4x^2 - 11x + 6$
17. $x^2 + 7xy + 12y^2$
18. $6x^2 - 17xy + 5y^2$
19. $6a^2 + 13ab - 5b^2$
20. $3x^2 + 4xy - 4y^2$
21. $x^2 - 18x + 32$
22. $y^2 - 4yz - 12z^2$

Factor the following expressions using the difference of two squares.

23. $9x^2 - 25$
24. $2a^4 - 2c^4$
25. $x^4 - 16y^4$
26. $y^4 - 81$

Factor the following expressions using the sum/difference of two cubes.

27. $27x^3 - 1$
28. $a^3 + 8$
29. $8a^3 + 125$
30. $x^3y^3 - 27z^3$
31. $a^9 - b^9$
32. $x^3y^6 + 8z^9$

Factor the following expressions using substitution.

33. $(y - 2)^2 + 5(y - 2) - 36$
34. $(x + 3)^2 + 8(x + 3) + 12$
35. $4(m - n)^2 - 28(m - n) - 32$
36. $(a + b)^2 - a - b - 12$
37. $(2x - y)^2 - (x + y)^2$
38. $4(a + b)^2 - (a - b)^2$
39. $9(3x + 1)^2 - (x - 3)^2$
40. $(a - 2b)^2 - (a + 2b)^2$

Factor the following expressions using several methods.

41. $16x^{12} + 2x^3$
42. $x^2(x^2 - 9) + 2x(x^2 - 9) - 15(x^2 - 9)$
43. $2x^2(a^2 - b^2) + x(a^2 - b^2) - 3a^2 + 3b^2$
44. $x^2(x^2 - 25) - 25(x^2 - 25)$

Factor the following expressions.

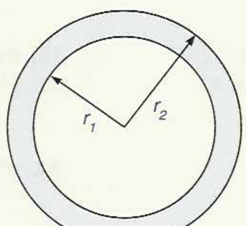
45. $m^2 - 49$
46. $81 - x^2$
47. $x^2 + 6x + 5$
48. $a^2 + 11a + 10$
49. $7a^2 + 36a + 5$
50. $3x^2 + 13x + 4$
51. $2a^2 + 15a + 18$
52. $5b^2 + 16b + 12$
53. $a^2b^2 + 2ab - 8$
54. $x^2y^2 - 5xy - 14$
55. $27a^3 + b^3$
56. $x^3 + 64y^3$
57. $25x^2(3x + y) + 5x(3x + y)$
58. $36x^2(2a - b) - 12x(2a - b)$
59. $10x^2 - 20xy + 10y^2$
60. $6a^2 - 24ab - 48b^2$
61. $4m^2 - 16n^2$
62. $9x^2 - 36y^2$
63. $(a - b)^2 - (2x + y)^2$
64. $(3a + b)^2 - (x - y)^2$
65. $3x^6 - 81y^3$
66. $32a^4 - 4ab^9$
67. $12x^3y^2 - 30x^2y^3 + 18xy^4$
68. $9x^5y - 6x^3y^3 + 3x^2y^2$
69. $4x^2 - 36y^2$
70. $36 - a^2b^4$
71. $3ax - 2by - bx + 6ay$
72. $6am - 3an + 4bm - 2bn$
73. $27a^9 - b^3c^3$
74. $x^{12} - y^3z^6$
75. $5a^2 - 32a - 21$
76. $7a^2 + 16a - 15$
77. $a^4 - 5a^2 + 4$
78. $x^4 - 37x^2 + 36$
79. $4a^2 - 4ab - 15b^2$
80. $6x^2 + 7xy - 3y^2$
81. $y^4 - 16$
82. $a^4 - 81$
83. $4a^2 + 10a + 4$
84. $6x^2 + 18x - 60$
85. $(x + y)^2 - 8(x + y) - 9$
86. $(a - 2b)^2 + 7(a - 2b) + 10$
87. $6a^2 + 7a - 5$
88. $4x^2 + 17x - 15$
89. $4ab(x + 3y) - 8a^2b^2(x + 3y)$
90. $3x^2y(m - 4n) + 15xy^2(m - 4n)$
91. $4a^2 - 20ab + 25b^2$
92. $9m^2 - 30mn + 25n^2$
93. $80x^5 - 5x$
94. $3b^5 - 48b$
95. $3a^5b - 18a^3b^3 + 27ab^5$
96. $3x^3y^3 + 6x^2y^4 + 3xy^5$
97. $9a^2 - (x + 5y)^2$
98. $7x(a^2 - 4b^2) + 14(a^2 - 4b^2)$
99. $3x^2 + 8x - 91$
100. $3x^2 - 32x - 91$
101. $24x^5y^9 + 81x^2z^6$

102. $a^8b^4 + 27a^2b^7$

103. $a^2(9 - x^2) - 6a(9 - x^2) - 9(x^2 - 9)$

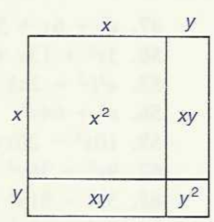
104. $x^2(16 - b^2) - 10x(b^2 - 16) + 25(16 - b^2)$

105. The area of a circle is $A = \pi r^2$, where r is the radius. The area of an annular ring with inner radius r_1 and outer radius r_2 would therefore be $\pi r_2^2 - \pi r_1^2$. Factor this expression. (If a computer program were being written to compute the area of many annular rings, the factored form would be far more efficient and, under certain circumstances, more accurate. See the discussion at the beginning of this section for the reasons why this is true.)



106. A freely falling body near the surface of the earth falls a distance $s = \frac{1}{2}gt^2$ in time t. The difference of two such distances measured for two different times would therefore be $\frac{1}{2}gt_2^2 - \frac{1}{2}gt_1^2$. Factor this expression.

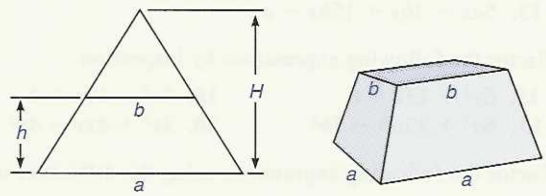
107. Multiplication and factoring of expressions sometimes have a geometric interpretation. For example, since the area of a rectangle is the product of its two dimensions, the square shown represents $(x + y)^2 = x^2 + 2xy + y^2$. Construct a rectangle that would represent $2x^2 + 7xy + 3y^2 = (2x + y)(x + 3y)$.



108. (See problem 107.) Construct a square that represents $(x - y)^2 = x^2 - 2xy + y^2$.

109. Factor $x^6 - 1$ two different ways. First, as a difference of two squares and second as a difference of two cubes. Draw a conclusion about the expression $x^4 + x^2 + 1$.

110. If the top of a pyramid with a square base is cut off parallel to the base, the resulting figure is called a *frustum* (the solid shown in the figure).



The volume of a frustum can be determined in the following manner.

The volume of a pyramid is $\frac{1}{3}hA$, where h is the height of the pyramid and A is the area of its base. This can be used to find the volume of a frustum: find the volume of the entire pyramid and subtract the volume at the top, which is missing. This missing volume is also a pyramid. Use this idea to show that the volume of a frustum with square base of length a, with height h, and with top horizontal dimension b (see the figure) is $\frac{1}{3}h(a^2 + ab + b^2)$. Use the following hints.

The volume of the entire pyramid shown is $\frac{1}{3}a^2H$. The volume of the top, missing, pyramid is $\frac{1}{3}b^2(H - h)$. The required volume V is the difference between these two values, or[13]

$$V = \tfrac{1}{3}[a^2H - b^2(H - h)].$$

Also, the geometric property of proportion between similar figures allows us to conclude that $\dfrac{a}{H} = \dfrac{b}{H - h}$. Use this to solve for H, and substitute the resulting expression for H in the formula for V. This will leave the formula for V using only the variables a, b, and h.

[13]A problem that correctly finds the volume of the frustum of a pyramid is found on an Egyptian papyrus in the Moscow Museum of Fine Arts. The papyrus is at least 3,600 years old.

111. Write a calculator or computer program that will compute the greatest common factor of two integers. The greatest common factor of two integers is the largest integer that evenly divides into both integers. One method of finding the greatest common factor of two integers is "Euclid's method." It is described as follows:

[1] Let x,y be two integers, $x > y$.
[2] Let d be the integer result of x/y.
[3] Let $r = x - dy$.
[4] If $r = 0$, the greatest common factor is y, and the process stops.
[5] Otherwise, replace x by y, and replace y by r.
[6] Proceed to the second step.

$\boxed{\textbf{112.}}$ Write a calculator or computer program that will compute the integer coefficients for the factors of quadratic trinomials. For example, for the trinomial $3x^2 + 10x - 8$, the program would provide the coefficients 1, 4, 3, and -2, which means that $3x^2 + 10x - 8 = (x + 4)(3x - 2)$. This is a *hard* problem. Some hints for one solution follow.

Assume you are given $Ax^2 + Bx + C$. The object is to find a, b, c, and d so that $(ax + b)(cx + d)$. First, find two integers m and n such that $mn = AC$ and $m + n = B$. Under these conditions, $Ax^2 + Bx + C$ can be rewritten $Ax^2 + mx + nx + C$. Then a is the greatest common factor of A and m, and b is the greatest common factor of n and C. Also, the sign of b is the same as the sign of n. Further, c is A/a, and d is C/b. See problem 111 for a discussion of finding the greatest common factor of two integers.

Skill and review

1. For what value(s) of x is $2x - 5$ equal to 0?

2. For what value(s) of x is $x^2 + 4$ negative or zero?

3. $-\frac{-1}{2}$ is the same as **a.** $\frac{1}{-2}$ **b.** $-\frac{1}{2}$ **c.** $\frac{1}{2}$ **d.** 2

4. Reduce $\frac{3x^3}{6x^6}$.

5. Multiply $\frac{3}{4} \cdot \frac{12}{5}$.

6. Add $\frac{2}{3} + \frac{1}{4}$.

7. Simplify $(2x - 3)(x + 1) - (x - 2)(x - 1)$.

8. Divide $\frac{5}{8} \div 2$.

1–4 Rational expressions

> If one printer can print x pages per hour and another can print $x + 2$ pages per hour, then the combined rate for these printers is $\frac{1}{x} + \frac{1}{x + 2}$. Combine this expression.

This problem illustrates the fact that many physical situations are represented by expressions in which a variable appears in the denominator. In this section we examine expressions that are fractions in which the numerator and denominator are polynomials. This is reflected in the following definition.

> **Rational expression**
>
> A rational expression is an expression of the form $\frac{P}{Q}$, where P and Q are polynomials and $Q \neq 0$.

The following are rational expressions: $\dfrac{2x}{3}$, $\dfrac{3x-2}{x^2-5x-9}$, and $\dfrac{x-2}{x+8}$.

Reducing rational expressions

Just as the fraction $\frac{2}{8}$ can be reduced to $\frac{1}{4}$, so many rational expressions can also be reduced. This is done by using the fundamental principle of rational expressions.

> **Fundamental principle of rational expressions**
>
> If P, Q, and R are polynomials, $Q \neq 0$, and $R \neq 0$, then $\dfrac{P \cdot R}{Q \cdot R} = \dfrac{P}{Q}$.

This principle states that factors that are common to the numerator and denominator of a rational expression (R) may be eliminated. When all common factors except 1 or -1 are eliminated, we say the rational expression is *reduced to its lowest terms.*

We need to comment about the signs of rational expressions. A useful principle is that the following are equivalent:

$$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$$

■ ***Example 1–4 A***

Reduce each rational expression to its lowest terms.

1. $\dfrac{6x^5y}{8x^3y} = \dfrac{3x^2}{4}$ — Reduce by GCF $2x^3y$

2. $\dfrac{2x-6}{2x^2-x-15} = \dfrac{2(x-3)}{(x-3)(2x+5)}$ — Factor

   $= \dfrac{2}{2x+5}$ — Reduce by GCF $(x-3)$

3. $-\dfrac{3-x}{x^2-9} = -\dfrac{-(x-3)}{(x-3)(x+3)}$ — $3 - x = -(x-3)$

   $= -\dfrac{-1}{x+3} = \dfrac{1}{x+3}$ — $-\dfrac{-a}{b} = \dfrac{a}{b}$ ■

Just as we saw with polynomials (section 1–2), there is an arithmetic of rational expressions. We will see how to perform addition, subtraction, multiplication, and division with rational expressions.

Multiplication and division of rational expressions

Multiplication and division of rational expressions are done in the same way as with fractions in arithmetic.

Multiplication of rational expressions

If P, Q, R, and S are polynomials, $Q \neq 0$, and $S \neq 0$, then

$$\frac{P}{Q} \cdot \frac{R}{S} = \frac{PR}{QS}$$

Division of rational expressions

If P, Q, R, and S are polynomials, $Q \neq 0$, $R \neq 0$, $S \neq 0$, then

$$\frac{P}{Q} \div \frac{R}{S} = \frac{P}{Q} \cdot \frac{S}{R}$$

The terms $\frac{R}{S}$ and $\frac{S}{R}$ are called **reciprocals** of each other. To divide by an expression is therefore equivalent to multiplying by the reciprocal of the divisor.

■ ***Example 1–4 B***

Find the indicated products or quotients. Assume all denominators represent nonzero numbers.

1. $\frac{3x - 1}{x - 4} \cdot \frac{x - 4}{3x^2 + 14x - 5}$

$= \frac{(3x - 1)(x - 4)}{(x - 4)(3x - 1)(x + 5)}$ Factor; $\frac{P}{Q} \cdot \frac{R}{S} = \frac{PR}{QS}$

$= \frac{1}{x + 5}$ Reduce by $(x - 4)$, $(3x - 1)$

2. $\frac{x^2 + 2x}{5x} \div (2x^2 + x - 6)$

$= \frac{x^2 + 2x}{5x} \cdot \frac{1}{2x^2 + x - 6}$ $\frac{P}{Q} \div \frac{R}{S} = \frac{P}{Q} \cdot \frac{S}{R}$; $a = \frac{a}{1}$

$= \frac{x^2 + 2x}{5x(2x^2 + x - 6)} = \frac{x(x + 2)}{5x(2x - 3)(x + 2)} = \frac{1}{5(2x - 3)}$ ■

Addition and subtraction of rational expressions

Two properties govern addition and subtraction of rational expressions.

Addition/subtraction of rational expressions

If P, Q, R, and S are polynomials, $Q \neq 0$, and $S \neq 0$, then

$$[1] \quad \frac{P}{Q} + \frac{R}{Q} = \frac{P+R}{Q} \text{ and } \frac{P}{Q} - \frac{R}{Q} = \frac{P-R}{Q}$$

Concept

If two rational expressions have the same denominator they may be added or subtracted by adding or subtracting the numerators only and retaining the denominator.

$$[2] \quad \frac{P}{Q} + \frac{R}{S} = \frac{PS+QR}{QS} \text{ and } \frac{P}{Q} - \frac{R}{S} = \frac{PS-QR}{QS}$$

Concept

If two rational expressions have different denominators they may be added or subtracted by "cross multiplying" (see the following discussion).

Property [1] is the definition of addition and subtraction of rational expressions, and is used when adding or subtracting rational expressions with the same denominator. Property [2], sometimes called "cross multiplying," is a short cut for adding/subtracting rational expressions with different denominators. (We saw the same principle in section 1–1 for real numbers.) It avoids having to find the least common denominator. Property [2] can be remembered as a sequence of three multiplications, which are shown graphically as $\frac{P}{Q} \times \frac{R}{S}$ (1, 2, 3). Observe that the first product should be PS, not QR; this is necessary in the case of subtraction. A demonstration that property [2] is valid is left as an exercise.

■ ***Example 1–4 C***

Add or subtract.

1. $\dfrac{3x-1}{x-3} - \dfrac{5x+4}{x-3} = \dfrac{(3x-1)-(5x+4)}{x-3}$ $\qquad \dfrac{P}{Q} - \dfrac{R}{Q} = \dfrac{P-R}{Q}$

$= \dfrac{-2x-5}{x-3} = -\dfrac{2x+5}{x-3}$

Note A common error is to omit the parentheses in expressions where subtraction is involved, as in $\dfrac{3x - 1 - 5x + 4}{x-3}$. This error produces the wrong result.

2. $\dfrac{3x}{2a} + \dfrac{5y}{3b} = \dfrac{3x(3b) + 2a(5y)}{2a(3b)}$

$= \dfrac{9bx + 10ay}{6ab}$

$\dfrac{P}{Q} + \dfrac{R}{S} = \dfrac{PS + QR}{QS}$
(cross multiply)

3. $\dfrac{3x - 1}{x - 2} - \dfrac{5x + 4}{x - 3}$

$= \dfrac{(3x - 1)(x - 3) - (x - 2)(5x + 4)}{(x - 2)(x - 3)}$

$\dfrac{P}{Q} - \dfrac{R}{S} = \dfrac{PS - QR}{QS}$
(cross multiply)

$= \dfrac{-2x^2 - 4x + 11}{x^2 - 5x + 6}$ ■

The properties above serve well when the least common denominator of the rational expressions is their product. However, in other cases it pays to find the least common denominator first and convert each expression to one having this as its denominator. The **least common denominator** (LCD) of two or more rational expressions is the smallest expression into which each denominator will divide. For example, for the fractions $\frac{5}{12}$, $\frac{3}{20}$, and $\frac{7}{30}$, the LCD is 60, since 12, 20, and 30 all divide evenly into 60, but they do not divide into any smaller number. The following rules describe how to find the LCD of two or more rational expressions when we cannot determine this by inspection.

To find the LCD of two or more rational expressions:
- Factor each denominator completely. The resulting factors are called **prime factors.**
- Write the product of each prime factor.
- Apply the greatest exponent of each factor found in the previous step.

Note that when an integer is factored completely, the factors are called **prime numbers.** These are the natural numbers that are divisible (evenly) only by one and itself. The first few prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, ... There are an infinite number of prime numbers.

■ ***Example 1–4 D***

Add or subtract.

1. $\dfrac{3}{2x^2} - \dfrac{5}{6x} + \dfrac{5}{4xy}$

To find the LCD we first factor each denominator completely:

$$\left.\begin{aligned} 2x^2 &= 2x^2 \\ 6x &= 2 \cdot 3x \\ 4xy &= 2^2xy \end{aligned}\right\} \text{The prime factors are 2, 3, } x\text{, and } y.$$

The highest exponent of the factor 2 is 2; for 3 it is 1; for x it is 2; for y the highest exponent is 1.

$2 \cdot 3xy$	Form the product of the prime factors
$2^2 \cdot 3x^2y$	Apply the largest exponent of each prime factor
$12x^2y$	Simplify

Now multiply both the numerator and denominator of each fraction by the factor that makes that denominator the LCD:

$$\frac{3}{2x^2} \cdot \frac{6y}{6y} - \frac{5}{6x} \cdot \frac{2xy}{2xy} + \frac{5}{4xy} \cdot \frac{3x}{3x}$$

$$= \frac{18y}{12x^2y} - \frac{10xy}{12x^2y} + \frac{15x}{12x^2y}$$

$$= \frac{18y - 10xy + 15x}{12x^2y}$$

2. $\dfrac{2x}{x+2} + \dfrac{4}{x^2-4} - \dfrac{2}{x^2-4x+4}$

$$= \frac{2x}{x+2} + \frac{4}{(x+2)(x-2)} - \frac{2}{(x-2)^2}$$

The LCD is $(x+2)(x-2)^2$.

$$= \frac{2x}{x+2} \cdot \frac{(x-2)^2}{(x-2)^2} + \frac{4}{(x+2)(x-2)} \cdot \frac{x-2}{x-2} - \frac{2}{(x-2)^2} \cdot \frac{x+2}{x+2}$$

$$= \frac{2x^3 - 8x^2 + 8x}{(x+2)(x-2)^2} + \frac{4x-8}{(x+2)(x-2)^2} - \frac{2x+4}{(x+2)(x-2)^2}$$

$$= \frac{2x^3 - 8x^2 + 8x + (4x-8) - (2x+4)}{(x+2)(x-2)^2}$$

$$= \frac{2x^3 - 8x^2 + 8x + 4x - 8 - 2x - 4}{(x+2)(x-2)^2}$$

$$= \frac{2x^3 - 8x^2 + 10x - 12}{x^3 - 2x^2 - 4x + 8}$$ ■

Complex rational expressions

A complex rational expression is one in which the numerator or denominator is itself a rational expression. Such expressions can be simplified in either of two ways.

■ ***Example 1–4 E***

Simplify the complex rational expression $\dfrac{\dfrac{6}{a^2} - 4}{6 - \dfrac{2}{a}}$.

Method 1: Perform the indicated division.

$$\frac{\frac{6}{a^2} - 4}{6 - \frac{2}{a}}$$

$$= \left(\frac{6}{a^2} - \frac{4}{1}\right) \div \left(\frac{6}{1} - \frac{2}{a}\right) \qquad 4 = \frac{4}{1};\ 6 = \frac{6}{1}$$

$$= \frac{6(1) - 4a^2}{1(a^2)} \div \frac{6a - 2(1)}{1(a)} \qquad \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$$

$$= \frac{6 - 4a^2}{a^2} \cdot \frac{a}{6a - 2} \qquad \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

$$= \frac{a(6 - 4a^2)}{a^2(6a - 2)}$$ Multiply numerators and denominators

$$= \frac{2a(3 - 2a^2)}{2a^2(3a - 1)}$$ Factor

$$= \frac{3 - 2a^2}{a(3a - 1)}$$

Method 2: Multiply the numerator and denominator by the LCD of $\frac{6}{a^2}$ and $\frac{2}{a}$, which is a^2.

$$\frac{\left(\frac{6}{a^2} - 4\right) \cdot a^2}{\left(6 - \frac{2}{a}\right) \cdot a^2}$$

$$= \frac{\frac{6}{a^2} \cdot \frac{a^2}{1} - 4 \cdot a^2}{6 \cdot a^2 - \frac{2}{a} \cdot \frac{a^2}{1}}$$ Distributive axiom

$$= \frac{6 - 4a^2}{6a^2 - 2a}$$

$$= \frac{2(3 - 2a^2)}{2a(3a - 1)}$$

$$= \frac{3 - 2a^2}{a(3a - 1)}$$ ■

Mastery points

Can you

- Reduce rational expressions to lowest terms?
- Perform addition, subtraction, multiplication, and division involving rational expressions?
- Simplify complex rational expressions?

Exercise 1–4

Reduce each rational expression to lowest terms.

1. $\dfrac{24p^5q^7}{18pq^3}$
2. $\dfrac{-27mn^3}{36m^4n}$
3. $\dfrac{4a+12}{3a+9}$
4. $\dfrac{-36x}{42x^3+24x}$
5. $\dfrac{a^2-9}{4a+12}$
6. $\dfrac{6a+3b}{4a^2-b^2}$
7. $\dfrac{64-49p^2}{7p-8}$
8. $\dfrac{m^2-4m-12}{m^2-m-6}$
9. $\dfrac{6a^3-6b^3}{a^2-b^2}$
10. $\dfrac{2a^2-3a+1}{2a^2+a-1}$
11. $\dfrac{8-2a}{a^3-64}$
12. $\dfrac{6x^2+17x+7}{12x^2+13x-35}$
13. $\dfrac{3a^2+16a-12}{6-7a-3a^2}$
14. $\dfrac{x^2+3x-10}{x^3-8}$

Perform the indicated additions and subtractions.

15. $\dfrac{3x}{2y}-\dfrac{2y}{5x}$
16. $\dfrac{a-2b}{3a}+\dfrac{5a-b}{2b}$
17. $\dfrac{x}{x+1}-\dfrac{3-x}{x-1}$
18. $\dfrac{2x-1}{x+3}-\dfrac{5x-1}{x+3}$
19. $\dfrac{3x-5}{x-4}+\dfrac{3x+2}{4-x}$
20. $\dfrac{3}{4}-\dfrac{2x}{3x+5}$
21. $\dfrac{3}{5}-\dfrac{2}{a}+\dfrac{9}{2b}$
22. $\dfrac{x-1}{x}-\dfrac{x-1}{x^2+x+1}$

Perform the indicated operations.

23. $\dfrac{12b}{7a}\cdot\dfrac{28a^3}{4b^3}$
24. $\dfrac{a-6}{6a+18}(a+3)$
25. $\dfrac{2}{x-3}+\dfrac{5}{x^2-3x}$
26. $\dfrac{5x}{x^2-4}-\dfrac{3}{x-2}$
27. $\dfrac{-6a}{a^2-a-6}-\dfrac{7a}{a^2+7a+10}$
28. $\dfrac{12}{5y-10}+\dfrac{7}{2y+4}$
29. $\dfrac{2a-1}{2a-3}+\dfrac{4-a}{3-2a}$
30. $\dfrac{x+2}{x-8}-\dfrac{x-6}{8-x}$
31. $\dfrac{4a+8}{3a-12}\cdot\dfrac{a-2}{5a+10}$
32. $\dfrac{4x^2-49}{8x^3+27}\cdot\dfrac{4x^2+12x+9}{2x^2-13x+21}$
33. $(x^2-2x-3)\div\dfrac{4x-4}{x^2-1}$
34. $\dfrac{r+4}{r^2-1}\div\dfrac{r^2-16}{r+1}$
35. $\dfrac{3y}{5y-10}+\dfrac{19}{2y+4}$
36. $\dfrac{10}{4a-6}-\dfrac{13}{3a+9}$
37. $\dfrac{3y}{y^2+5y+6}-\dfrac{5}{4-y^2}$
38. $\dfrac{3}{6b^2-4bc}-\dfrac{4}{6c^2-9bc}$
39. $\dfrac{m^2-9}{3m+4}\cdot\dfrac{9m^2-16}{m^2+6m+9}$
40. $\dfrac{a^2+2a+1}{1-4a}\cdot\dfrac{16a^2-1}{a^2-1}$
41. $\dfrac{x^2-25}{2x+10}\div(x^2-10x+25)$
42. $(4x^2-9)\div\dfrac{4x+6}{x+3}$

Simplify each complex rational expression.

43. $\dfrac{\frac{1}{3}+\frac{1}{2}}{\frac{2}{3}-\frac{1}{6}}$ **44.** $\dfrac{3-\frac{1}{5}}{5+\frac{3}{10}}$ **45.** $\dfrac{1+\dfrac{1}{a}}{2-\dfrac{3}{ab}}$ **46.** $\dfrac{\dfrac{2}{a}-\dfrac{3}{b}}{\dfrac{5}{ab}+\dfrac{3}{2a}}$

47. $\dfrac{\dfrac{3x-4}{8}}{\dfrac{2x+1}{10}}$ **48.** $\dfrac{\dfrac{2}{3a+2}}{\dfrac{4}{3a-2}}$ **49.** $\dfrac{\frac{2}{3}-\frac{1}{2}}{\frac{2}{3}+\frac{1}{2}}$ **50.** $\dfrac{2-\frac{3}{5}}{5+\frac{1}{5}}$

51. $\dfrac{\dfrac{m^2-n^2}{n}}{\dfrac{1}{n}-\dfrac{1}{m}}$ **52.** $\dfrac{y-\dfrac{x}{y}}{x-\dfrac{y}{x}}$ **53.** $\dfrac{5-\dfrac{3}{x+2}}{3+\dfrac{2}{x+2}}$ **54.** $\dfrac{\dfrac{6}{x^2-x}-2}{\dfrac{3}{x-1}+2}$

55. $\dfrac{\dfrac{1}{x-1}-\dfrac{2}{x+1}}{\dfrac{6}{x+1}-\dfrac{3}{x-1}}$ **56.** $\dfrac{\dfrac{x}{x-4}+\dfrac{3}{x}}{5-\dfrac{9}{x^2-4x}}$ **57.** $\dfrac{\dfrac{a}{a-b}+\dfrac{a}{a+b}}{\dfrac{3}{a^2-b^2}}$ **58.** $\dfrac{\dfrac{2}{x}-\dfrac{3}{x+y}}{\dfrac{2}{x}+\dfrac{3}{x+y}}$

59. In electricity theory the following expression arises: $\dfrac{\dfrac{V_1}{R_1}+\dfrac{V_2}{R_2}+\dfrac{V_3}{R_3}}{\dfrac{1}{R_1}+\dfrac{1}{R_2}+\dfrac{1}{R_3}}$. Simplify this complex fraction.

60. Simplify this complex fraction from electricity theory: $\dfrac{\dfrac{V_1}{R_1}+\dfrac{V_2}{R_2}}{\dfrac{1}{R_1}+\dfrac{1}{R_2}}$.

61. When making a round trip whose one-way distance is d, the average rate (speed) traveled, r, is $\dfrac{2d}{\dfrac{d}{r_1}+\dfrac{d}{r_2}}$, where r_1 and r_2 are the average rates for each direction of the trip. Simplify this expression.

62. If one printer can print x pages per hour and another can print $x+2$ pages per hour, then the combined rate for these printers is $\dfrac{1}{x}+\dfrac{1}{x+2}$. Combine this expression.

63. If x is the speed of a boat in still water (in knots), and $x+3$ is the speed of the boat in a 3 knot current, then the difference in times it will take to cover one knot under these two different conditions is $\dfrac{1}{x}-\dfrac{1}{x+3}$. Combine this expression.

64. If two investments have a rate of return of r_1 and r_2, with $r_1 > r_2$, then the difference in time it would take each investment to produce an amount of interest I on a principal P is $\dfrac{I}{Pr_2}-\dfrac{I}{Pr_1}$. Combine these expressions.

65. The following were presented as properties of addition and subtraction of rational expressions:

$$\frac{P}{Q}+\frac{R}{S}=\frac{PS+QR}{QS} \text{ and } \frac{P}{Q}-\frac{R}{S}=\frac{PS-QR}{QS}.$$

Prove that these are true by computing

$$\frac{P}{Q}+\frac{R}{S} \text{ and } \frac{P}{Q}-\frac{R}{S},$$

first obtaining a common denominator.

66. In studying the reliability of mechanical systems, MTBF means mean (average) time between failures. If a system is made up of a series of three devices with MTBF of MTBF_1, MTBF_2, and MTBF_3, then the MTBF for the system, MTBF_S, is

$$\text{MTBF}_S=\left(\frac{1}{\text{MTBF}_1}+\frac{1}{\text{MTBF}_2}+\frac{1}{\text{MTBF}_3}\right)^{-1}.$$

Rewrite so the three rational expressions are combined and the negative exponent is removed.

67. See the previous problem for terminology. A certain computer contains three main sections: CPU (central processor unit), power supply, and disk memory. Suppose the CPU has a MTBF of 1,000 hours, the power supply has a MTBF of 3,000 hours, and the disk memory has a MTBF of 1,800 hours. Since the failure of any of these parts causes the failure of the computer, the series formula of problem 66 applies for finding the MTBF of the computer. Find this value to the nearest hour for this computer.

68. In pulmonary function testing in medicine, pulmonary compliance is the volume change per unit of pressure change for the lungs (C_L), the thorax (C_T), or the lungs-thorax system (C_{LT}). The compliance for all three is recorded in liters per centimeter of water. These values are related by $\frac{1}{C_L} + \frac{1}{C_T} = \frac{1}{C_{LT}}$. Find an expression for C_{LT}.

69. Write a program for a computer or programmable calculator that will determine if a given natural number is prime. The program either prints out the first prime factor of the number or the word "Prime."

Since no even number after two is prime (why?), and even numbers are easy to recognize (how?) it is only necessary to test odd natural numbers. The easiest method is to simply divide the number by all odd integers beginning with three. It is only necessary to go as far as the square root of the number.

By way of example, to determine whether 323 is prime we could divide by 3, 5, 7, 9, . . . , 17, since $\sqrt{323} \approx 17.97$.

Skill and review

1. Compute **a.** $\sqrt{25}$ **b.** $\sqrt{100}$ **c.** $\sqrt{400}$
2. Compute **a.** $\sqrt[3]{8}$ **b.** $\sqrt[3]{64}$ **c.** $\sqrt[6]{64}$
3. Compute **a.** $\sqrt{4 \cdot 9}$ **b.** $\sqrt{4} \cdot \sqrt{9}$
4. Multiply $2x^2y^3(2^3x^2y)$.
5. Find the smallest integer that is greater than or equal to the given integer and is divisible by 4.

   Example: 10 The answer is 12 because 12 is divisible by 4 but 10 and 11 are not.

   Example: 37 The answer is 40 because 40 is divisible by 4 but 37, 38, and 39 are not.

   a. 7 **b.** 8 **c.** 13 **d.** 44 **e.** 54
6. Consider $81a^4b^8 = 3a^2b^5(3^xa^yb^z)$. Find x, y, and z.

1–5 Radicals

> The advertisement for an apartment says it has an enormous square living room with over 900 square feet. What is the length and width of the room?

The area of a square is the square of its length. For this living room the length must be about 30 feet, because 30^2 is 900. We could say that $\sqrt{900}$ (the square root of 900) is 30.

Definition of a radical

Consider the symbol $\sqrt{9}$. It means "the square root of 9"; it is "the number that, when squared, gives 9." This number is 3. (Note that -3 squared also gives 9, however.) Similarly, $\sqrt[3]{8}$ means "the cube root of 8"; this is 2, since $2^3 = 8$. We will generalize this idea in the following definition.[14]

[14]The $\sqrt{}$ symbol, now called the radix symbol, was introduced by Christoff Rudolff in 1526. The more general notation $\sqrt[n]{}$ was used in 1891 by Giuseppe Peano.

> **Principal *n*th root of *a***
>
> Let $a, b \in R$, a and b have the same sign (or are both zero), $n \in N$. Then $\sqrt[n]{a} = b$ means $b^n = a$. b is called the principal nth root of a.
>
> ***Concept***
>
> b is the number whose nth power is a; if $a > 0$ then $b > 0$.

Note We require b and a to have the same sign so that b is positive when n is even; this makes $\sqrt{9}$ be 3 and not -3, for example.

We read $\sqrt[n]{a}$ "the principal nth root of a" although we often omit the word "principal" for convenience. $\sqrt[n]{a}$ is called a **radical,** n is the **index,** and a is the **radicand.** When the index is 2 it is omitted; this is called the square root. When the index is 3 this is called the cube root. The rest are verbalized as the fourth root, fifth root, etc.

■ ***Example 1–5 A***

Find the indicated root.

1. $\sqrt[3]{8} = 2$ $\quad 2^3 = 8$
2. $\sqrt[4]{625} = 5$ $\quad 5^4 = 625$
3. $\sqrt[3]{-27} = -3$ $\quad (-3)^3 = -27$
4. $-\sqrt{25} = -5$ $\quad -\sqrt{25} = -(\sqrt{25}) = -5$
5. $\sqrt{-25}$ Not a real number

There is no real value b such that $b^2 = -25$ (since any real number squared produces a nonnegative value). Thus, $\sqrt{-25}$ is not a real number. For the same reason, *when the radicand is negative and the index is even, the radical does not represent a real number.* We will see how to deal with these cases when we examine the system of complex numbers in section 1–7.

The nth root of most real numbers is an irrational number and can only be approximated. This can be done with a calculator, which will be illustrated in the next section, after discussing rational exponents.

Simplifying radicals

We will state what it means to simplify a radical later in this section, but first we examine specific cases where radicals can be written more simply.

Whenever the index of a radical is equal to the exponent of the radicand the following property can be applied.

> ***n*th root of *n*th power property**
>
> $$\sqrt[n]{a^n} = \begin{cases} |a| & \text{if } n \text{ is even} \\ a & \text{if } n \text{ is odd} \end{cases}$$
>
> In particular $\sqrt{x^2} = |x|$.

Note If $a \geq 0$ then $\sqrt[n]{a^n} = a$ whether n is even or odd, since in this case $|a| = a$.

Thus, $\sqrt[3]{(-2)^3} = -2$ — n is odd so $\sqrt[n]{a^n} = a$

but $\sqrt{(-2)^2} = 2$ — n is even so $\sqrt[n]{a^n} = |a|$

This property can also be used to simplify a radical like $\sqrt[3]{8x^3y^6}$. The radicand can be viewed as a cube by rewriting as $\sqrt[3]{(2xy^2)^3}$. The nth root of nth power property then tells us that this is $2xy^2$.

■ ***Example 1–5 B***

Simplify. Assume the variables are nonnegative in 1 through 3.

1. $\sqrt{5^6} = \sqrt{(5^3)^2} = 5^3 = 125$

2. $\sqrt{81a^8b^2c^4} = \sqrt{9^2a^8b^2c^4}$ — Rewrite 81 as 9^2
$= \sqrt{(9a^4bc^2)^2} = 9a^4bc^2$

3. $\sqrt[5]{32a^5b^{10}c^{20}} = \sqrt[5]{2^5a^5b^{10}c^{20}}$
$= \sqrt[5]{(2ab^2c^4)^5} = 2ab^2c^4$

In the following examples, variables may represent negative values. Thus, we use $\sqrt{x^2} = |x|$.

4. $\sqrt{4x^2y^4} = \sqrt{(2xy^2)^2} = |2xy^2| = 2y^2|x|$

5. $\sqrt{x^2 - 4x + 4} = \sqrt{(x-2)^2} = |x-2|$

6. $\sqrt{16a^4b^8} = \sqrt{(4a^2b^4)^2} = |4a^2b^4|$
$= 4a^2b^4$ — $a^2 \geq 0$, $b^4 \geq 0$

Multiplication and division of radicals, and more on simplification

The procedure illustrated above applies when the index of the radical divides evenly into the exponent of each factor of the radicand. However, the index of the radical may not divide evenly into each exponent of the radicand; in this case, the following property can be used. It applies *when the exponent of any factor of the radicand is greater than or equal to the index.*

> **Product property of radicals**
> If $a \geq 0$, $b \geq 0$, and $n \varepsilon N$, then $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$.

The product property tells us how to multiply radicals as well as how to simplify certain radicals.

■ ***Example 1–5 C***

Simplify. Assume all variables are nonnegative.

1. $\sqrt{8a^5b^8} = \sqrt{2^3a^5b^8}$

Observe that the index 2 does not divide into all of the exponents; we can use the product rule for radicals as follows.

$= \sqrt{2^2 \cdot 2 \cdot a^4ab^8}$ — $2^3 = 2^2 \cdot 2$; $a^5 = a^4 \cdot a$

$= \sqrt{2^2a^4b^8}\sqrt{2a}$ — Product rule for radicals

We have factored the radical into two radicals; in the first, each exponent is divisible by the index, 2, and in the second, each exponent is less than the index.

$$= \sqrt{(2a^2b^4)^2}\sqrt{2a}$$
$$= 2a^2b^4\sqrt{2a} \qquad n\text{th root of } n\text{th power}$$

2. $\sqrt{32} = \sqrt{2^5} = \sqrt{2^4}\sqrt{2} = 2^2\sqrt{2} = 4\sqrt{2}$

3. $\sqrt[3]{54x^7y^2} = \sqrt[3]{2 \cdot 3^3x^6 \cdot xy^2}$ $\qquad 54 = 2 \cdot 27 = 2 \cdot 3^3;\ x^7 = x^6 \cdot x$

$= \sqrt[3]{3^3x^6}\sqrt[3]{2xy^2} = 3x^2\sqrt[3]{2xy^2}$

4. $\sqrt[4]{96a^5b^8c^{15}} = \sqrt[4]{2^5 \cdot 3a^5b^8c^{15}} = \sqrt[4]{2^4a^4b^8c^{12}}\sqrt[4]{2 \cdot 3ac^3} = 2ab^2c^3\sqrt[4]{6ac^3}$

Perform the indicated multiplications, then simplify. Assume all variables are nonnegative.

5. $(2\sqrt{3})(5\sqrt{27})$

$= (2 \cdot 5)(\sqrt{3})(\sqrt{27})$ $\qquad (ab)(cd) = (ac)(bd)$

$= 10\sqrt{3 \cdot 27}$ $\qquad$ Product rule for radicals

$= 10\sqrt{81} = 10 \cdot 9 = 90$

6. $\sqrt[3]{4x^2y^4}\sqrt[3]{2xy}$

$\sqrt[3]{(4x^2y^4)(2xy)} = \sqrt[3]{8x^3y^5} = 2xy\sqrt[3]{y^2}$ ■

The following property can help simplify radicals involving rational expressions as well as tell how to perform some division operations with radicals.

Quotient property of radicals

If $a \geq 0$, $b > 0$, and $n \in N$, then $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$.

The quotient property will be illustrated in the following example, along with the following property.

Index/exponent common factor property

Given $\sqrt[n]{a^m}$ and $a \geq 0$. If m and n have a common factor, this factor can be divided from m and from n.

Concept

If the index and every exponent of a radicand can be reduced by some common factor, this reduction is valid.

For example, given $\sqrt[6]{x^4y^2}$, $x \geq 0$, $y \geq 0$. Since the value 2 is a common factor for every exponent and the index, we can rewrite this radical by dividing every exponent and the index by 2, obtaining $\sqrt[3]{x^2y}$.

Before illustrating these properties we will now make clear what we mean by ''simplifying'' a radical.

> A radical is simplified if
> 1. The exponent of all factors of the radicand is less than the index.
> 2. There is no radical in a denominator.
> 3. There is no fraction under the radical.
> 4. The index of the radical and the power of each factor in the radicand have no common factor other than 1.

■ ***Example 1–5 D***

Simplify. Assume all variables are nonnegative and do not represent division by zero.

1. $\sqrt{\dfrac{16a^4b^5c}{25d^8}} = \dfrac{\sqrt{2^4a^4b^5c}}{\sqrt{5^2d^8}}$ — We have a fraction in a radical

$= \dfrac{2^2a^2b^2\sqrt{bc}}{5d^4} = \dfrac{4a^2b^2\sqrt{bc}}{5d^4}$ — Quotient rule for radicals

2. $\dfrac{6x}{\sqrt[3]{4x}}$ — We have a radical in a denominator

In this case we eliminate a radical in a denominator by multiplying it (and the numerator) by a factor that will make all of the exponents multiples of the index (in this case 3).

$= \dfrac{6x\sqrt[3]{2x^2}}{\sqrt[3]{2^2x}\sqrt[3]{2x^2}}$ — We want each exponent under the radical in the denominator to be a multiple of 3

$= \dfrac{6x\sqrt[3]{2x^2}}{\sqrt[3]{8x^3}} = \dfrac{6x\sqrt[3]{2x^2}}{2x} = 3\sqrt[3]{2x^2}$

3. $\dfrac{6}{\sqrt{3}}$ — For square roots multiply the numerator and denominator by the radical in the denominator

$$\frac{6\cdot\sqrt{3}}{\sqrt{3}\cdot\sqrt{3}} = \frac{6\sqrt{3}}{\sqrt{3^2}} = \frac{6\sqrt{3}}{3} = 2\sqrt{3}$$

4. $\dfrac{6}{\sqrt[3]{3}}$ $\quad \dfrac{6}{\sqrt[3]{3}}\cdot\dfrac{\sqrt[3]{3^2}}{\sqrt[3]{3^2}} = \dfrac{6\sqrt[3]{3^2}}{\sqrt[3]{3^3}} = \dfrac{6\sqrt[3]{9}}{3} = 2\sqrt[3]{9}$

5. $\sqrt[5]{\dfrac{3a^6}{64b^5c^4}} = \dfrac{\sqrt[5]{3a^6}}{\sqrt[5]{2^6b^5c^4}}$ — Quotient rule for radicals

$= \dfrac{a\sqrt[5]{3a}}{2b\sqrt[5]{2c^4}}$ — Simplify the radicals in the numerator and denominator

$= \dfrac{a\sqrt[5]{3a}\sqrt[5]{2^4c}}{2b\sqrt[5]{2c^4}\sqrt[5]{2^4c}}$ — $2\cdot 2^4 = 2^5$; $c^4\cdot c = c^5$

$= \dfrac{a\sqrt[5]{2^4(3)ac}}{2b\sqrt[5]{2^5c^5}} = \dfrac{a\sqrt[5]{48ac}}{2b(2c)} = \dfrac{a\sqrt[5]{48ac}}{4bc}$

6. $\sqrt[4]{ab} \cdot \sqrt[4]{ab^3}$
$= \sqrt[4]{a^2b^4}$ — Multiply
$= \sqrt{ab^2}$ — Since the index (4) and each exponent are divisible by 2, reduce them all by dividing each by 2. Recall that we do not usually show the index 2 in a square root. ■

Addition and subtraction with radicals—combining like terms

The usual rules for manipulating expressions, along with the properties illustrated above, provide the means of simplifying expressions involving radicals. The product and quotient properties of radicals tell us how to multiply and divide with radicals. *We add and subtract radicals by combining like terms,* as with any algebraic expression. For example,

$$5\sqrt{6} + 2\sqrt{6} = 7\sqrt{6}$$

just as

$$5a + 2a = 7a$$

■ ***Example 1–5 E***

Perform the indicated operations. Assume all variables are nonnegative.

1. $7\sqrt{2} + 3\sqrt{8} - \sqrt{32}$
$= 7\sqrt{2} + 3(2\sqrt{2}) - 4\sqrt{2}$ — $\sqrt{8} = \sqrt{4}\sqrt{2} = 2\sqrt{2}$ and
$= 7\sqrt{2} + 6\sqrt{2} - 4\sqrt{2}$ — $\sqrt{32} = \sqrt{16}\sqrt{2} = 4\sqrt{2}$
$= 9\sqrt{2}$ — Combine like terms

2. $\sqrt[3]{81x^4y} - x\sqrt[3]{24xy}$
$= 3x\sqrt[3]{3xy} - 2x\sqrt[3]{3xy}$ — Simplify each radical
$= x\sqrt[3]{3xy}$ — Combine like terms

3. $5\sqrt{2}(3\sqrt{2} - 6\sqrt{3})$
$= 5\sqrt{2} \cdot 3\sqrt{2} - 5\sqrt{2} \cdot 6\sqrt{3}$ — $a(b - c) = ab - ac$
$= 15(2) - 30\sqrt{6}$ — $\sqrt{2}\sqrt{2} = 2$; $\sqrt{2}\sqrt{3} = \sqrt{6}$
$= 30 - 30\sqrt{6}$

4. $(\sqrt{2x} - \sqrt{6})(\sqrt{2x} + 3\sqrt{6})$
$= 2x + 3\sqrt{12x} - \sqrt{12x} - 3(6)$ — $(a - b)(c + d)$ $= ac + ad - bc - bd$
$= 2x + 2\sqrt{12x} - 18$
$= 2x + 4\sqrt{3x} - 18$ ■

Many expressions happen to have two terms in the denominator in which one or both expressions involve square roots. *These radicals can be eliminated from the denominator by multiplying the numerator and denominator by the conjugate of the denominator.* This relies on the fact that $(a - b)(a + b) = a^2 - b^2$; $a - b$ and $a + b$ are conjugates (section 1–3), and, if a or b is a square root, then neither a^2 nor b^2 will contain a square root. This process of rationalizing the denominators is illustrated in the following examples.

Example 1–5 F

Rationalize the denominators. Assume all variables are nonnegative and no denominator equals zero.

1. $\dfrac{8}{\sqrt{11}-3}$

The conjugate of the denominator is $\sqrt{11}+3$

$$= \frac{8}{\sqrt{11}-3}\cdot\frac{\sqrt{11}+3}{\sqrt{11}+3}$$
$$= \frac{8(\sqrt{11}+3)}{(\sqrt{11})^2-3^2}$$

$(a-b)(a+b)=a^2-b^2$

$$= \frac{8(\sqrt{11}+3)}{11-9} = \frac{8(\sqrt{11}+3)}{2}$$
$$= 4(\sqrt{11}+3)$$

2. $\dfrac{\sqrt{ab}+\sqrt{b}}{\sqrt{a}+\sqrt{b}}$

The conjugate of the denominator is $\sqrt{a}-\sqrt{b}$

$$= \frac{\sqrt{ab}+\sqrt{b}}{\sqrt{a}+\sqrt{b}}\cdot\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}}$$
$$= \frac{\sqrt{a^2b}-\sqrt{ab^2}+\sqrt{ab}-b}{(\sqrt{a})^2-(\sqrt{b})^2}$$
$$= \frac{a\sqrt{b}-b\sqrt{a}+\sqrt{ab}-b}{a-b}$$

■

In advanced applications we often come across expressions like those shown in the following example.

Example 1–5 G

Perform the indicated operations and simplify.

1. $\dfrac{\sqrt{3}}{2}\cdot\dfrac{1}{\sqrt{5}}-\left(-\dfrac{1}{2}\right)\left(\dfrac{3}{\sqrt{5}}\right)$

$$= \frac{\sqrt{3}}{2\sqrt{5}}+\frac{3}{2\sqrt{5}} = \frac{\sqrt{3}+3}{2\sqrt{5}} = \frac{\sqrt{3}+3}{2\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{15}+3\sqrt{5}}{10}$$

2. $\sqrt{\dfrac{1-\dfrac{\sqrt{3}}{2}}{2}}$

$$= \sqrt{\frac{1}{2}\left(1-\frac{\sqrt{3}}{2}\right)} = \sqrt{\frac{1}{2}\left(\frac{2}{2}-\frac{\sqrt{3}}{2}\right)} = \sqrt{\frac{2-\sqrt{3}}{4}} = \frac{\sqrt{2-\sqrt{3}}}{2}$$

■

Mastery points

Can you

- Find exact values of indicated roots?
- Simplify radicals using the nth root of nth power property?
- Simplify radicals using the product and quotient properties?
- Perform algebraic operations on expressions which contain radicals?
- Rationalize denominators?

Exercise 1–5

Find the indicated root.

1. $\sqrt{289}$ **2.** $\sqrt{625}$ **3.** $\sqrt[5]{32}$ **4.** $\sqrt[4]{81}$ **5.** $\sqrt[3]{-125}$ **6.** $\sqrt[3]{125}$

Simplify the following radical expressions. Do not assume that variables represent nonnegative real numbers.

7. $\sqrt{4x^2}$ **8.** $\sqrt{9x^4}$ **9.** $\sqrt{25x^6y^8}$ **10.** $\sqrt{\dfrac{1}{4a^4}}$

11. $\sqrt{\dfrac{16x^6}{9y^{10}}}$ **12.** $\sqrt{4x^6(x-3)^4}$ **13.** $\sqrt{x^4-6x^2+9}$ **14.** $\sqrt{\dfrac{x^2}{x^2+12x+36}}$

Simplify the following radical expressions. Assume that all variables represent nonnegative real numbers.

15. $\sqrt[3]{40}$ **16.** $\sqrt[4]{32}$ **17.** $\sqrt{200}$ **18.** $\sqrt[3]{8{,}000}$

19. $\sqrt[5]{64a^7b^5}$ **20.** $\sqrt{50x^6y^7z^2}$ **21.** $\sqrt{200x^2y^9z^{12}}$ **22.** $\sqrt[3]{52a^4b^5c^6}$

23. $\sqrt[4]{16x^3y^6}$ **24.** $\sqrt[4]{625a^6b^9c^4}$ **25.** $\sqrt[3]{a^4}\sqrt[3]{a}\sqrt[3]{a^7}$ **26.** $\sqrt[4]{4a^2b}\sqrt[4]{4a^2b^6}$

27. $\sqrt[3]{25a^2b^4c}\sqrt[3]{25a^2b^5c^2}$ **28.** $\sqrt[3]{12x^3y^2z^5}\sqrt[3]{12x^2yz^2}$ **29.** $\dfrac{8}{\sqrt{2}}$ **30.** $\dfrac{9}{4\sqrt{3}}$

31. $\dfrac{12a}{\sqrt{50a}}$ **32.** $\dfrac{5x}{\sqrt{20x}}$ **33.** $\sqrt{\dfrac{8}{27}}$ **34.** $\sqrt{\dfrac{7}{18}}$

35. $\sqrt[3]{\dfrac{8}{9}}$ **36.** $\sqrt[3]{\dfrac{3}{20}}$ **37.** $\sqrt{\dfrac{12x^5y^6}{5z}}$ **38.** $\sqrt{\dfrac{x^{14}y^7}{27z^3w}}$

39. $\sqrt[5]{\dfrac{64x^{14}y^6}{x^2yz^4w^8}}$ **40.** $\sqrt[3]{\dfrac{a^5b}{4ab^3c}}$ **41.** $\sqrt[3]{\dfrac{16a^5}{b^7c^2}}$ **42.** $\sqrt[4]{\dfrac{x^4y^9}{8z^2}}$

43. $\dfrac{\sqrt[4]{16x^6y^5}}{\sqrt[4]{27x^4y^3z^6}}$ **44.** $\dfrac{\sqrt[3]{5x^7y^2}}{\sqrt[3]{30xy^7z^3}}$ **45.** $\dfrac{\sqrt[4]{8x^5y^7}}{\sqrt[4]{50x^3y^9}}$ **46.** $\dfrac{\sqrt[3]{5a^2b^6}}{\sqrt[3]{10a}}$

Simplify the following radical expressions. Assume that all variables represent nonnegative real numbers.

47. $4\sqrt{2}-8\sqrt{2}+\sqrt{8}$ **48.** $\sqrt{5}-6\sqrt{5}+\sqrt{45}$ **49.** $5\sqrt{3}+7\sqrt{12}-\sqrt{75}$

50. $2\sqrt{8}-\sqrt{50}+3\sqrt{2}$ **51.** $2\sqrt{48}-3\sqrt{27}$ **52.** $5\sqrt{2x^3}-2x\sqrt{8x}$

53. $\sqrt[3]{16}+5\sqrt[3]{24}$ **54.** $2b\sqrt[4]{32b}-\sqrt[4]{162b^5}$ **55.** $-2\sqrt{36a^2b}+4\sqrt{25a^2b}$

56. $3\sqrt{18xy^2}-2y\sqrt{2x}$ **57.** $2\sqrt{3a}(\sqrt{3a}-2\sqrt{27a})$ **58.** $3\sqrt{x}(2\sqrt{xy}-\sqrt{x})$

59. $\sqrt[3]{4x}(\sqrt[3]{2x^2}+\sqrt[3]{4x}-\sqrt[3]{x^2})$ **60.** $(\sqrt{2}-3)(\sqrt{2}+3)$ **61.** $(3-4\sqrt{x})(4-2\sqrt{x})$

62. $(\sqrt{8a}-\sqrt{2})(\sqrt{2a}-\sqrt{8})$ **63.** $(5\sqrt{3}-2\sqrt{6})(\sqrt{3}+\sqrt{12})$ **64.** $\sqrt{2xy}(\sqrt{2xy}-\sqrt{2x}+\sqrt{6y})$

65. $(2\sqrt{2x^2}-\sqrt{4x})^2$ **66.** $(\sqrt{5}-\sqrt{10})^2$

Rationalize the denominators. Assume that all variables represent nonnegative real numbers.

67. $\dfrac{\sqrt{5}}{\sqrt{5}-\sqrt{2}}$ **68.** $\dfrac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}}$ **69.** $\dfrac{a+\sqrt{b}}{a-\sqrt{b}}$

70. $\dfrac{\sqrt{6}}{\sqrt{6}-12}$ **71.** $\dfrac{\sqrt{2x}}{\sqrt{6x}+\sqrt{2}}$ **72.** $\dfrac{2a}{\sqrt{a}+\sqrt{ab}}$

Perform the indicated operations and simplify.

73. $\dfrac{1}{3}\cdot\dfrac{\sqrt{7}}{5}-\dfrac{\sqrt{2}}{3}\cdot\dfrac{\sqrt{2}}{5}$ **74.** $\dfrac{1}{\sqrt{5}}\left(-\dfrac{\sqrt{5}}{4}\right)+\dfrac{3}{\sqrt{5}}\left(-\dfrac{1}{4}\right)$ **75.** $\dfrac{2}{\sqrt{5}}\cdot\dfrac{1}{\sqrt{3}}-\dfrac{3}{\sqrt{5}}\cdot\dfrac{2}{\sqrt{3}}$

76. $-\dfrac{5}{2\sqrt{2}}\left(-\dfrac{1}{\sqrt{2}}\right)-\dfrac{1}{2}\left(\dfrac{\sqrt{5}}{3}\right)$ **77.** $\sqrt{\dfrac{3-\dfrac{1}{\sqrt{2}}}{2}}$ **78.** $\sqrt{\dfrac{4+\dfrac{\sqrt{3}}{2}}{2}}$

79. $\sqrt{\dfrac{4-\dfrac{\sqrt{5}}{8}}{2}}$ **80.** $\sqrt{\dfrac{1-\dfrac{\sqrt{3}}{\sqrt{5}}}{5}}$

Simplify the following expressions. Assume all variables are nonnegative.

81. $\sqrt[4]{a^6}$ **82.** $\sqrt[6]{x^4y^2}$ **83.** $\sqrt[10]{32x^5y^{10}}$ **84.** $\sqrt[8]{a^8b^{12}c^{16}}$

85. If $x \geq 0$ and $y \geq 0$, then $x - y$ can be factored into $(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})$ by viewing it as the difference of two squares. Similarly, viewed as the difference of two cubes, $x - y = (\sqrt[3]{x}-\sqrt[3]{y})Q$, where Q represents a quadratic expression in the variables $\sqrt{x}$ and $\sqrt{y}$.

a. Determine what expression is represented by Q.

b. Factor $x + y$ using a similar strategy with the factorization of a sum of two cubes.

c. Factor $8x - y$, first viewing it as a difference of two cubes, and then viewing it as a difference of two squares.

86. Use the factorization for $a^3 + b^3$ as a guide to a way to rationalize the denominator of the fraction $\dfrac{3xy}{\sqrt[3]{x}+\sqrt[3]{y}}$. That is, use the fact that

$$a + b = (\sqrt[3]{a})^3 + (\sqrt[3]{b})^3 = (\sqrt[3]{a}+\sqrt[3]{b})(\ .\ .\ .\ ).$$

87. Using problem 86 as a hint, rationalize the denominator of the fraction $\dfrac{\sqrt[3]{x}}{\sqrt[3]{2x^2}-\sqrt[3]{3x}}$.

88. Using problems 86 and 87 as a hint, rationalize $\dfrac{3}{\sqrt[4]{x}-\sqrt[4]{y}}$. Assume $x > 0$ and $y > 0$.

89. Show that $\sqrt{\dfrac{2-\sqrt{3}}{2+\sqrt{3}}} = \dfrac{1-\dfrac{1}{\sqrt{3}}}{1+\dfrac{1}{\sqrt{3}}}$.

90. About 3,000 years ago the Babylonians developed the following method for computing an approximate value for a square root. Today it is known as Newton's method.

Let $x = \sqrt{a}$ be the desired value. Let a_1 be a first approximation to $\sqrt{a}$. Let $b_1 = \dfrac{a}{a_1}$. If $a_1 = b_1$, then since $a_1b_1 = a$, $a_1 = \sqrt{a}$. Of course this is unlikely to happen. If a_1 is too small then b_1 is too large, and vice versa. This means that their average will be a better approximation to the root. Thus, let $a_2 = \dfrac{a_1+b_1}{2}$ and $b_2 = \dfrac{a}{a_2}$. Again, let $a_3 = \dfrac{a_2+b_2}{2}$ and $b_3 = \dfrac{a}{a_3}$ etc. This method can be continued indefinitely, until a_i is sufficiently accurate.

With a calculator, use this method to compute $\sqrt{19}$ to 4-digit accuracy using only the operations of addition, subtraction, multiplication, and division. Since $4 < \sqrt{19} < 5$, use 4.5 for a_1, and note how many iterations are necessary to achieve the desired accuracy. $\sqrt{19} \approx 4.359$.

91. There is a property of radicals that allows us to rewrite $\sqrt{\sqrt[3]{5}}$ using only one radical.

a. Guess what this one radical might be, then try to demonstrate that your guess is correct.

b. Deduce what this property of radicals is, in general (always assuming variables to be nonnegative for simplicity).

Skill and review

1. Add $\frac{1}{3} + \frac{3}{4}$.
2. Multiply $\frac{1}{3} \cdot \frac{3}{4}$.
3. Simplify $3a^{-2}$.
4. Simplify $\sqrt{9a^4b^6}$, $a,b \geq 0$.
5. Compute $\frac{1}{\sqrt[3]{-8}}$.
6. Simplify $\frac{8a^8}{2a^2}$.
7. Simplify $\frac{-2a^{-2}}{8a^8}$.

1–6 Rational exponents

A formula which relates the leg diameter D_l necessary to support a body with body length L_b for large vertebrates is $D_l = cL_b^{1.5}$, where c is a constant depending on the vertebrate in question. Suppose body length increases by a factor of 4. What needs to happen to leg diameter to support the body?

The expression that is the right member of the equation above contains the exponent 1.5, which is not an integer. In this section we learn how to interpret and use exponents that are not simply integers.

Meaning of integer exponents

Recall that we have defined the meaning of expressions with integer exponents.

If $n \in N$, then $x^n = n$ factors of x.
If $x \neq 0$, then $x^0 = 1$.
If $n \in N$ and $n > 0$, then $x^{-n} = \frac{1}{x^n}$.

Thus, $9^2 = 9 \cdot 9 = 81$, $9^0 = 1$, and $9^{-2} = \frac{1}{9^2} = \frac{1}{81}$.

We also noted that the following properties apply to integer exponents, assuming that $a,b \in R$ and $m,n \in J$ and that no variable represents zero where division by zero would be indicated.

[1] $a^m a^n = a^{m+n}$ [2] $\frac{a^m}{a^n} = a^{m-n}$ [3] $(ab)^m = a^m b^m$

[4] $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ [5] $(a^m)^n = a^{mn}$

Definition of rational exponents

It is natural to ask what might be the value of $9^{\frac{1}{2}}$. If we define this expression involving a rational exponent to have some meaning, we want to do this in a fashion that does not violate the properties of exponents above.

Since $(\sqrt{9})^2 = 9$, and we would expect $\left(9^{\frac{1}{2}}\right)^2 = 9^{\frac{1}{2}\cdot 2} = 9^1 = 9$, it is natural to define $9^{\frac{1}{2}}$ to mean $\sqrt{9}$. Thus, $9^{\frac{1}{2}} = \sqrt{9} = 3$. This reasoning leads to the following definition of an exponent of the form $\dfrac{1}{n}$, $n \in N$, $n > 1$.

Definition of $a^{\frac{1}{n}}$

If $a \in R$, $n \in N$, $n > 1$, and $\sqrt[n]{a} \in R$, then

$$a^{\frac{1}{n}} = \sqrt[n]{a}.$$

This definition is extended to rational exponents in which the numerator is not one[15] in the following definition.

Definition of $a^{\frac{m}{n}}$

If $m,n \in N$ and $\sqrt[n]{a} \in R$, then

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m.$$

The following property can also be shown to be true.

Equivalence of $\left(a^{\frac{1}{n}}\right)^m$ and $(a^m)^{\frac{1}{n}}$ for $a \geq 0$

If $a \geq 0$ and $m,n \in N$, then

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = (a^m)^{\frac{1}{n}}.$$

This last property is important because it allows us to rewrite expressions in whichever way is easier for us. For example, we can rewrite $9^{\frac{3}{2}}$ as $\left(9^{\frac{1}{2}}\right)^3$ (numerator of fraction outside the parentheses), which becomes $3^3 = 27$. We can also write something like $(\sqrt{7})^{\frac{2}{3}}$ as $((\sqrt{7})^2)^{\frac{1}{3}}$ (numerator of fraction inside the parentheses), which becomes $7^{\frac{1}{3}}$ or $\sqrt[3]{7}$.

[15] a^x where x is a negative integer or fraction was introduced in concept by John Wallis in 1656, but Newton introduced our modern notation in 1676.

We need one more definition before we are finished.

Definition of $a^{-\frac{m}{n}}$

If $a \neq 0$, $m,n \in N$, and $a^{\frac{m}{n}} \in R$, then

$$a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}}.$$

■ ***Example 1–6 A***

Rewrite each expression in terms of radicals. Simplify if possible. Assume all variables represent nonnegative values.

1. $(16x)^{\frac{1}{4}} = \sqrt[4]{16x}$ — Definition: $a^{\frac{1}{4}} = \sqrt[4]{a}$

$= \sqrt[4]{16}\sqrt[4]{x} = 2\sqrt[4]{x}$ — Simplify the radical

2. $8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2$ — Definition: $a^{\frac{2}{3}} = \left(a^{\frac{1}{3}}\right)^2$

$= (\sqrt[3]{8})^2$ — Definition: $a^{\frac{1}{3}} = \sqrt[3]{a}$

$= 2^2 = 4$

3. $(32x^{10})^{-\frac{3}{5}} = \dfrac{1}{(32x^{10})^{\frac{3}{5}}}$ — Definition of $a^{-\frac{m}{n}}$

$$= \frac{1}{(\sqrt[5]{32x^{10}})^3} = \frac{1}{(2x^2)^3} = \frac{1}{8x^6}$$ ■

If $a < 0$, $a^{\frac{m}{n}}$ is not defined whenever n is even, since in this case $\sqrt[n]{a}$ is not defined. Rational exponents that are not relatively prime may be reduced when both expressions are real numbers. By way of example, $(-8)^{\frac{5}{3}}$ is defined and has the value -32, but $(-8)^{\frac{10}{6}}$ is not defined since $\sqrt[6]{-8}$ is not real. It is true that *rational exponents may always be reduced when the base is nonnegative.*

Simplifying expressions with rational exponents

It can be shown that if the values of all bases are nonnegative then the properties of exponents [1]–[5] stated at the beginning of this section can be applied to rational exponents. Thus, for example, to multiply we add exponents, etc. This is illustrated in the following example.

■ ***Example 1–6 B***

Simplify using the properties of exponents [1]–[5] as well as the definitions and theorems of this section. Assume all variables represent nonnegative values.

1. $9^{\frac{3}{4}} \cdot 9^{\frac{3}{4}} = 9^{\frac{3}{4}+\frac{3}{4}} = 9^{\frac{6}{4}} = 9^{\frac{3}{2}} = (\sqrt{9})^3 = 3^3 = 27$

2. $\left(3x^{\frac{1}{2}}\right)\left(5x^{\frac{2}{3}}y\right) = 15x^{\frac{1}{2}+\frac{2}{3}}y = 15x^{\frac{7}{6}}y$ $\qquad a^m a^n = a^{m+n}$

3. $\left(27x^{\frac{1}{2}}y^4\right)^{\frac{1}{3}} = 27^{\frac{1}{3}}x^{\frac{1}{2}\cdot\frac{1}{3}}y^{4\cdot\frac{1}{3}} = 3x^{\frac{1}{6}}y^{\frac{4}{3}}$ $\qquad (ab)^m = a^m b^m$

4. $\dfrac{x^{-\frac{1}{2}}y^{\frac{2}{3}}}{4^{-\frac{1}{2}}x^2y^{-\frac{1}{3}}} = \dfrac{1}{4^{-\frac{1}{2}}} \cdot \dfrac{x^{-\frac{1}{2}}}{x^2} \cdot \dfrac{y^{\frac{2}{3}}}{y^{-\frac{1}{3}}}$ $\qquad$ Separate for clarity

$= 4^{\frac{1}{2}}x^{\left(-\frac{1}{2}-2\right)}y^{\frac{2}{3}-\left(-\frac{1}{3}\right)}$ $\qquad$ Definition of $a^{-\frac{m}{n}}$; $\dfrac{a^m}{a^n} = a^{m-n}$

$= 2x^{-\frac{5}{2}}y^1 = \dfrac{2y}{x^{\frac{5}{2}}}$ ■

Approximate values for rational exponents and radicals

As stated in the previous section most radicals are irrational. We can only obtain decimal approximations to these values. Many scientific/engineering calculators have a key marked $\boxed{\sqrt[x]{y}}$ designed to approximate roots. If a calculator does not have this key it probably has a key marked $\boxed{x^{1/y}}$, which is used the same way, since $\sqrt[y]{x} = x^{1/y}$ for $x \geq 0$. The Texas Instruments TI-81 has neither key. The value of $\sqrt[x]{y}$ is obtained by the sequence y $\boxed{\wedge}$ x $\boxed{x^{-1}}$ $\boxed{\text{ENTER}}$.

Some calculators do not have a $\boxed{\sqrt[x]{y}}$ or $\boxed{x^{1/y}}$ key. In this case the sequence $\boxed{x^y}$ "index" $\boxed{1/x}$ must be used,[16] which has the same effect as the $\boxed{x^{1/y}}$ key. This is illustrated in parts 3, 4, 5 of the next example.

The $\boxed{x^y}$ key is also the key to computing expressions directly expressed with rational exponents. The following examples illustrate various ways calculators compute numeric values of expressions with rational exponents.

■ ***Example 1–6 C***

Compute the following values to six digits of accuracy.

1. $5^{\frac{3}{4}}$ $\qquad$ 5 $\boxed{x^y}$ $\boxed{(}$ 3 $\boxed{\div}$ 4 $\boxed{)}$ $\boxed{=}$

$= 3.34370$ $\qquad$ TI-81: 5 $\boxed{\wedge}$ $\boxed{(}$ 3 $\boxed{\div}$ 4 $\boxed{)}$ $\boxed{\text{ENTER}}$

[16]Nonalgebraic calculators (Hewlett-Packard™, for example) use "index" $\boxed{1/x}$ $\boxed{y^x}$.

2. $(-8.2)^{-\frac{2}{3}}$

Calculators will not accept a negative base with the $\boxed{x^y}$ key. In this case it is necessary to determine the sign of the answer ourselves. To see the sign of this result, rewrite in radical form:

$$(-8.2)^{-\frac{2}{3}} = \left((-8.2)^{-\frac{1}{3}}\right)^2 = \left(\frac{1}{\sqrt[3]{-8.2}}\right)^2.$$

$\sqrt[3]{-8.2}$ is negative, but when squared the result is positive. Thus, we actually compute $8.2^{-\frac{2}{3}}$, since it is the same value:

$= 0.245918$ $8.2\ \boxed{x^y}\ \boxed{(}\ 2\ \boxed{\div}\ 3\ \boxed{)}\ \boxed{+/-}\ \boxed{=}$

TI-81: $8.2\ \boxed{\wedge}\ \boxed{(}\ \boxed{(-)}\ 2\ \boxed{\div}\ 3\ \boxed{)}\ \boxed{\text{ENTER}}$

3. $\sqrt[3]{21.6}$ $21.6\ \boxed{\sqrt[x]{y}}\ 3\ \boxed{=}$ or $21.6\ \boxed{x^y}\ 3\ \boxed{1/x}\ \boxed{=}$

$= 2.78495$ TI-81: $\boxed{\text{MATH}}\ 4\ 21.6\ \boxed{\text{ENTER}}$

4. $\sqrt[5]{2003^2}$ $2003\ \boxed{x^2}\ \boxed{\sqrt[x]{y}}\ 5\ \boxed{=}$ or $2003\ \boxed{x^2}\ \boxed{x^y}\ 5\ \boxed{1/x}\ \boxed{=}$

$= 20.9253$ TI-81: $2003\ \boxed{x^2}\ \boxed{\wedge}\ 5\ \boxed{x^{-1}}\ \boxed{\text{ENTER}}$ ■

Mastery points

Can you

- Rewrite expressions with rational exponents in terms of radicals?
- Simplify expressions with rational exponents using the properties of exponents?
- Evaluate indicated numeric roots?
- Compute approximate values to numeric expressions involving radicals and rational exponents?

Exercise 1–6

Rewrite each expression in terms of radicals. Simplify if possible. Assume all variables represent nonnegative values.

1. $64^{\frac{1}{4}}$ **2.** $16^{\frac{3}{4}}$ **3.** $8^{-\frac{2}{3}}$ **4.** $100^{-\frac{1}{2}}$ $\boxed{5.}$ $(-8)^{-\frac{2}{3}}$

6. $81^{-\frac{3}{4}}$ **7.** $(16x)^{\frac{1}{2}}$ **8.** $(50x^3)^{\frac{1}{2}}$ **9.** $(81x^3)^{\frac{1}{4}}$ **10.** $(27x^6)^{\frac{1}{4}}$

11. $(32x^6y^7)^{\frac{1}{4}}$ **12.** $100^{\frac{1}{4}}$ **13.** $(8x^3)^{\frac{1}{2}}$

Simplify.

14. $5^{\frac{3}{2}} \cdot 5^{\frac{1}{2}}$ **15.** $25^{\frac{1}{4}} \cdot 25^{\frac{1}{4}}$ **16.** $2b^{\frac{3}{4}}\left(3b^{\frac{1}{4}}\right)$ **17.** $2a^{\frac{1}{2}}\left(4a^{\frac{1}{4}}\right)$

18. $\left(a^{\frac{1}{3}}\right)^3$ **19.** $\left(b^{\frac{3}{4}}\right)^{\frac{8}{3}}$ **20.** $\left(x^{\frac{3}{4}}y^{\frac{3}{5}}\right)^{\frac{8}{3}}$ **21.** $\left(4x^{\frac{4}{7}}y^{\frac{6}{5}}z\right)^{\frac{1}{2}}$

22. $\left(\frac{2}{3}x^{\frac{2}{3}}y^{-\frac{1}{3}}\right)\left(3x^{\frac{1}{3}}y^{\frac{4}{3}}\right)$

23. $\left(\frac{1}{4}x^{\frac{1}{4}}y^{\frac{1}{2}}\right)\left(\frac{1}{2}x^{\frac{1}{4}}y^{\frac{1}{4}}\right)$

24. $\dfrac{x^{\frac{3}{4}}}{x^{\frac{1}{4}}}$

25. $\dfrac{ab^{\frac{1}{3}}}{a^2b^{\frac{5}{6}}}$

26. $\dfrac{x^{\frac{2}{3}}}{x}$

27. $\dfrac{a}{a^{\frac{1}{4}}}$

28. $\dfrac{4a^{\frac{1}{4}}b^{\frac{3}{2}}}{8a^{\frac{3}{4}}b^{\frac{1}{2}}}$

29. $\dfrac{6a^{\frac{1}{2}}}{2a^{\frac{1}{4}}}$

30. $\dfrac{4x^{\frac{-3}{4}}}{x^{\frac{1}{4}}y^{\frac{3}{4}}}$

31. $\dfrac{x^{\frac{5}{8}}y^{\frac{2}{3}}}{x^{\frac{3}{8}}y^{\frac{-2}{3}}}$

32. $\left(\dfrac{x^{\frac{-3}{5}}y^{\frac{3}{4}}}{z^{\frac{3}{10}}}\right)^{20}$

33. $\left(\dfrac{x^{\frac{4}{3}}y^{\frac{8}{5}}}{z^{-8}}\right)^{\frac{1}{8}}$

34. $\left(x^{\frac{a}{b}}y^{\frac{a}{c}}\right)^{\frac{bc}{a}}$

35. $\left(a^{\frac{m}{n}}b^{\frac{2}{n}}\right)^{2n}$

36. $\dfrac{3^{\frac{1}{n}}}{3^{\frac{3}{2n}}}$

37. $\left(\dfrac{2^{\frac{m+n}{m}}x^{\frac{n}{m}}}{2^{\frac{n}{m}}x^{\frac{n}{m}}y^{\frac{-n}{m}}}\right)^{m}$

Compute the following values to four decimal places of accuracy using your calculator.

38. $28^{\frac{1}{3}}$ **39.** $1{,}345^{\frac{3}{5}}$ **40.** $(-19)^{\frac{2}{3}}$ **41.** $(-200)^{\frac{1}{5}}$ **42.** $\sqrt[4]{96}$

43. $\sqrt[3]{-900}$ **44.** $\sqrt[6]{19^8}$ **45.** $\sqrt[4]{37.5^3}$ **46.** $(\sqrt[4]{5})^{\frac{2}{3}}$ **47.** $(\sqrt[5]{19})^{\frac{3}{5}}$

48. $(\sqrt[4]{97.1})^{\frac{5}{3}}$ **49.** $(\sqrt[3]{-500})^{\frac{4}{3}}$ **50.** $\sqrt[4]{19.5}$ **51.** $\sqrt[3]{0.125}$ **52.** $\sqrt[5]{0.825}$

53. $\sqrt[3]{18^7}$ **54.** $\sqrt[3]{0.5^4}$ **55.** $(\sqrt[5]{603.25})^4$ **56.** $(\sqrt[3]{31.5})^5$

57. A formula that relates the leg diameter D_l necessary to support a body with body length L_b for large vertebrates is $D_l = cL_b^{1.5}$, where c is a constant depending on the vertebrate in question. Suppose body length increases by a factor of 4. What needs to happen to leg diameter to support the body? *Hint:* Replace L_b by $4L_b$.

58. The volume of a sphere is $V = \frac{4}{3}\pi r^3$, where r is the radius. Thus, $r = \sqrt[3]{\dfrac{3V}{4\pi}}$. Since the cross section of a sphere is a circle, the area of the cross section at its widest part is $A = \pi r^2$, or in terms of the volume of the sphere, $A = \pi\left(\dfrac{3V}{4\pi}\right)^{2/3}$. Show that this can be transformed into $A = \dfrac{\sqrt[3]{\pi}}{4}(6V)^{2/3}$.

59. The formula $P = A\left[\dfrac{i}{1-(1+i)^{-N}}\right]$ gives the monthly payment on a fixed rate mortgage, where A is the amount borrowed, i is the monthly interest rate (yearly rate $\div$ 12), and N is the total number of monthly payments (number of years $\times$ 12). By way of example, if \$50,000 is borrowed at 9% yearly interest for 30 years, then $N = 30 \times 12 = 360$ and $i = 0.09/12 = 0.0075$, so the monthly payment is

$$P = 50{,}000\left(\frac{0.0075}{1 - 1.0075^{-360}}\right) = \$402.31.$$

Calculate the monthly payment on a loan of \$45,000 at 10% interest for 30 years.

60. (Refer to problem 59.) The formulas $B_n = A\left[\dfrac{i(1+i)^{n-1}}{(1+i)^N - 1}\right]$ and $I_n = P[1 - (1+i)^{n-1-N}]$ are related to monthly mortgage payments. If n is the nth monthly payment, then B_n is the amount of principal being paid that month, and I_n is the amount of interest being paid that month. N, A, and i are defined in problem 59. Find B_n and I_n for the first month of the loan in problem 59 (\$45,000, 10%, 30 years).

61. The following expression gives the value of what is called the second Fibonacci number (discussed further in chapter 12); compute this value:
$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^2-\left(\frac{1-\sqrt{5}}{2}\right)^2\right].$$

62. How can you find the fourth root of a number on a calculator using only the square root key? The eighth root?

63. Compute (with a calculator) the value of the expression: $\sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-\sqrt{5}}$.

64. Evaluate the expression $\dfrac{\dfrac{1}{\sqrt{3}}+\sqrt{\dfrac{2-\sqrt{3}}{2+\sqrt{3}}}}{1-\dfrac{1}{\sqrt{3}}\sqrt{\dfrac{2-\sqrt{3}}{2+\sqrt{3}}}}$ with a calculator.

Skill and review

1. Select the correct statement(s) about $\sqrt{-4}$.

   a. not real **b.** $= -2$ **c.** $= 2^{-1}$ **d.** $= \dfrac{1}{\sqrt{4}}$

2. Compute $(5i - 3)(2i + 7)$.
3. Compute **a.** $(-1)^5$ **b.** $(-1)^{20}$ **c.** $(-1)^{47}$
4. Compute $(\sqrt{2} - \sqrt{3})(3\sqrt{2} + \sqrt{6})$.
5. Rationalize the denominator of $\dfrac{2\sqrt{3}}{\sqrt{6}+\sqrt{2}}$.

1–7 Complex numbers

If two impedances Z_1 and Z_2 are connected in parallel in an electronic circuit, the total impedance, T, is related by the statement $T = \dfrac{Z_1 + Z_2}{Z_1 Z_2}$.

Here Z_1 and Z_2 are complex numbers. Numbers of this type have proved very useful in electronics theory. We study complex numbers in this section.

Definitions

For the last thousand years it was recognized that there are no real number solutions to the equation $x^2 = -4$. Nevertheless, such equations are often encountered in mathematics and the physical sciences (particularly electronics theory in physics).

Solutions to such equations would involve numbers like $\sqrt{-4}$, which cannot be a real number. This is because if $x = \sqrt{-4}$, then $x^2 = -4$. However $x^2 \geq 0$ for all real numbers. Thus, x could not be real.

The problem was solved several hundred years ago by extending the real number system in the way described by the definitions below. The result was the set of **complex numbers,** C. It is based on the idea of assuming that there is a square root of negative one. We call it the imaginary unit.

Imaginary unit i

There exists a number i such that $i = \sqrt{-1}$.

The imaginary unit[17] i was first used in 1777 by the famous mathematician Leonhard Euler. Engineers use the letter j for the same value. The imaginary unit is not part of the real number system.[18]

Square root of a negative real number

If b is a positive real number, then $\sqrt{-b} = i\sqrt{b}$.

This definition now gives us a way to write, for example, $\sqrt{-4}$:

$$\sqrt{-4} = i\sqrt{4} = 2i$$

Complex numbers are defined based on a notation suggested by W. R. Hamilton in 1837.

Complex numbers

The set of complex numbers = C =

$$\{a + bi \mid a, b \,\varepsilon\, R, \text{ and } i \text{ is the imaginary unit}\}$$

Concept

A complex number is a binomial; the first term is a real number and the second is the product of a real coefficient and the imaginary unit.

In $a + bi$, a is called the **real part** and b is called the **imaginary part.** For example, $3 - 8i$ is a complex number with real part 3 and imaginary part -8. An expression of the form $a + bi$ is said to be the **standard form** for a complex number. A complex number like $0 - 3i$ might be simply written as $-3i$, but it is nevertheless a complex number. We can also consider an expression such as 4 to represent $4 + 0i$, and thus view it as a complex number also. Equality of complex numbers is defined as follows.

Equality of complex numbers

Two complex numbers $a + bi$ and $c + di$ are equal if and only if $a = c$ and $b = d$.

Concept

Complex numbers are equal when their real parts and imaginary parts are each the same.

[17] H. Cardan was the earliest mathematician to seriously consider imaginary numbers. He demonstrated calculations with the square roots of negative numbers in his work *Ars magna* (Nurenberg, 1545).

[18] "I met a man recently who told me that, so far from believing in the square root of minus one, he did not even believe in minus one. This is at any rate a consistent attitude" (E. C. Titchmarsh).

Operations with complex numbers

Since $i = \sqrt{-1}$, $i^2 = -1$. Addition, subtraction, multiplication, and division of complex numbers are defined so that *we can use the algebra of the real number system, with the provision that i^2 be replaced by -1 wherever it appears.* The statement of the formal rules for addition, subtraction, multiplication, and division of complex numbers are left as exercises. (They are not necessary to perform computations by hand—they would be necessary if we wished to program a computer to perform these operations.[19])

■ Example 1–7 A

Perform the operations shown on the complex numbers.

1. $(5 - 3i) - (12 + 3i)$

$= 5 - 12 - 3i - 3i$ — Remove grouping symbols

$= -7 - 6i$ — Combine like terms

2. $(-2i)(8 - 3i)$

$= -16i + 6i^2$ — Multiply as with real-valued expressions

$= -16i + 6(-1)$ — $i^2 = -1$

$= -6 - 16i$ — Rewrite so real part is first

3. $(5 - 3i)(12 + 3i)$ — $(a - b)(c + d) = ac + ad - bc - bd$

$= 60 + 15i - 36i - 9i^2$

$= 60 - 21i - 9(-1)$ — $i^2 = -1$

$= 69 - 21i$ ■

Division is handled in a special way. The value $a - bi$ is called the **complex conjugate** of $a + bi$, and vice versa. *To divide by a complex number multiply by a fraction in which both the numerator and denominator are the complex conjugate of the divisor.* This works because, as will be seen below, the product of a complex number with its conjugate is a real number.

■ Example 1–7 B

Divide.

1. $\dfrac{2 - 3i}{5 + 2i}$

$= \dfrac{2 - 3i}{5 + 2i} \cdot \dfrac{5 - 2i}{5 - 2i}$ — Multiply by the conjugate of the denominator

$= \dfrac{10 - 4i - 15i + 6i^2}{25 - 10i + 10i - 4i^2}$

$= \dfrac{4 - 19i}{29}$ — The denominator is now the real number 29

$= \dfrac{4}{29} - \dfrac{19}{29}i$ — Put the answer in standard form $a + bi$

[19]The FORTRAN programming language is preprogrammed to recognize and use complex values. Some programmable calculators (TI-85 for example) will perform complex arithmetic.

2. $\dfrac{6-3i}{4i} = \dfrac{6-3i}{4i} \cdot \dfrac{-4i}{-4i}$ The conjugate of $0 + 4i$ is $0 - 4i$, or just $-4i$

$$= \frac{-24i + 12i^2}{-16i^2} = \frac{-12 - 24i}{16} = -\frac{3}{4} - \frac{3}{2}i$$ ■

Note 1. An answer such as $\dfrac{4 - 19i}{29}$ (part 1 example 1–7 B) is not complete. It should be written in the standard form (a binomial) $\dfrac{4}{29} - \dfrac{19}{29}i$.

2. The conjugate of a complex number such as $0 + bi$ is $0 - bi$, or simply $-bi$ (as in part 2 example 1–7 B).
3. Part 2 of example 1–7 B could be done by simply using $-i$ instead of $-4i$.

It is very important to avoid using the square roots of negative real values in calculations. Convert them to "i-notation" first. This is shown by the computation of $\sqrt{-2}\sqrt{-2}$. If we simply use the rules for radicals, we obtain

$$\sqrt{-2}\sqrt{-2} = \sqrt{(-2)(-2)} = \sqrt{4} = 2$$

whereas if we proceed with i-notation we obtain

$$\sqrt{-2}\sqrt{-2} = i\sqrt{2} \cdot i\sqrt{2} = i^2\sqrt{4} = -2$$

The second value is correct, not the first. The first is wrong because we assume the rules for radicals, such as $\sqrt{a}\sqrt{b} = \sqrt{ab}$, apply when a or b is negative. This is not true.[20]

■ *Example 1–7 C*

Simplify.

1. $\sqrt{-18}\sqrt{-6}$

$= i\sqrt{18} \cdot i\sqrt{6}$ Rewrite in i-notation before any computation

$= 3i\sqrt{2} \cdot i\sqrt{6} = 3i^2\sqrt{12}$

$= -3(2)(\sqrt{3}) = -6\sqrt{3}$

2. $\dfrac{\sqrt{-3} - 4}{\sqrt{-2} - 1}$

$= \dfrac{i\sqrt{3} - 4}{i\sqrt{2} - 1} \cdot \dfrac{i\sqrt{2} + 1}{i\sqrt{2} + 1}$ Multiply numerator and denominator by conjugate of the denominator

$= \dfrac{i^2\sqrt{6} + i\sqrt{3} - 4i\sqrt{2} - 4}{i^2(2) + i\sqrt{2} - i\sqrt{2} - 1}$

$= \dfrac{-\sqrt{6} - 4 + i\sqrt{3} - 4i\sqrt{2}}{-3}$ $i^2 = -1$; separate the real and imaginary terms in the numerator

[20] Mistakes of this type were made even by famous mathematicians as late as the eighteenth century.

$$= \frac{-(\sqrt{6}+4)}{-3} + \frac{i(\sqrt{3}-4\sqrt{2})}{-3} \qquad \text{Break up into real and imaginary parts}$$
$$= \frac{\sqrt{6}+4}{3} - \frac{\sqrt{3}-4\sqrt{2}}{3}i$$

The value of i^n, n a whole number, can only have one of four values. Computation shows that

$$i^1 = i$$
$$i^2 = -1$$
$$i^3 = -i \qquad i^3 = i^2 \cdot i = -i$$
$$i^4 = 1 \qquad i^4 = (i^2)^2 = (-1)^2$$

For values of n above four, the result is always one of the four values obtained above. For powers above four, *all multiples of four can be eliminated,* since they correspond to factors of one (see example 1–7 D).

■ Example 1–7 D

Compute the value of each expression.

Problem **Solution**

1. i^{43}

$i^{43} = i^{40} \cdot i^3 = 1 \cdot i^3 = i^3 = i^2 \cdot i$
$= (-1) \cdot i = -i \qquad i^2 = -1$

2. Evaluate $5x^7 - 3x^5 + x^4 - 6x^3 + 4$ for $x = i$.

$5i^7 - 3i^5 + i^4 - 6i^3 + 4$
$5i^3 - 3i + 1 - 6i^3 + 4 \qquad i^7 = i^3 \text{ and } i^5 = i$
$-i^3 - 3i + 5$
$-(-i) - 3i + 5 \qquad i^3 = -i$
$5 - 2i$

Mastery points

Can you

- Simplify the square root of a negative number?
- Add, subtract, multiply, and divide complex numbers?

Exercise 1–7

Perform the operations shown on the complex numbers.

1. $(-3 + 5i) - (2 - 3i)$
2. $(6 + 2i) + (6 - 12i)$
3. $(5 - 3i) - i + 4(2 + 3i)$
4. $-5(4 + 2i) - (5 + 3i) + 2i$
5. $(-8 + 3i)(2 - 7i)$
6. $2i(6 - 4i)$
7. $(5 - 3i)(5 + 3i)$
8. $(2 + i)(2 - i)$
9. $(5 - 2i)^2$
10. $i(2 + 5i)(\frac{1}{2} + 4i)$
$\boxed{11.}$ $i[(5 - 3i)(-2 + 4i) - (2 - i)^2]$
12. $[(3 - i) - (9 + 2i)][(2 + 3i)(2 - 3i)]$

Divide.

13. $\dfrac{3-4i}{2+5i}$ **14.** $\dfrac{8+2i}{4-i}$ **15.** $\dfrac{6+4i}{6-4i}$ **16.** $\dfrac{5-i}{i}$ **17.** $\dfrac{6i}{2-3i}$ **18.** $\dfrac{4-3i}{2-3i}$

Simplify the following expressions.

19. $\sqrt{-6}\sqrt{-12}$ **20.** $\sqrt{-8}\sqrt{-4}$ **21.** $\sqrt{5}\sqrt{-10}$ **22.** $\sqrt{-12}\sqrt{8}$

23. $(8-\sqrt{-8})-(-5+\sqrt{-50})$ **24.** $(2+3\sqrt{-20})+(\sqrt{-45}-3\sqrt{-80})$

25. $(3-\sqrt{-3})(4+\sqrt{-3})$ **26.** $\sqrt{-10}(\sqrt{-6}-4\sqrt{-8})$ **27.** $(2+\sqrt{-6})(3-\sqrt{-2})$

28. $(3\sqrt{-8}+\sqrt{2})(\sqrt{-2}-4\sqrt{8})$ **29.** $\dfrac{4-\sqrt{-6}}{2+3\sqrt{-2}}$ **30.** $\dfrac{5+2\sqrt{-14}}{4-\sqrt{-6}}$

31. $\dfrac{3-4\sqrt{-8}}{i(2-3\sqrt{-2})}$ **32.** $(2-\sqrt{-2})^4$ **33.** $\dfrac{\sqrt{-6}+\sqrt{6}}{\sqrt{-2}-\sqrt{2}}$ **34.** $\dfrac{2\sqrt{30}+\sqrt{-6}}{\sqrt{-12}}$

Compute the value of each expression.

35. i^{10} **36.** i^{21} **37.** i^{15} **38.** i^{19} **39.** i^{-5} **40.** i^{-3} **41.** i^{-15} **42.** i^{-2}

43. Evaluate $5x^9-6x^6+4x^5-2x^3+12x^2-1$ for $x=i$.

44. Evaluate $(2x^5-3x^2)(x^3+2x)$ for $x=i$.

In the following problems compute the value of the expression if $Z_1=2-i$, $Z_2=9+2i$, and $Z_3=5-3i$.

45. $\dfrac{Z_1Z_2}{Z_1-Z_2}$ **46.** $(Z_2-Z_1)(Z_2+Z_1)$ **47.** $Z_1^3-3Z_1^2+Z_1$ **48.** $\dfrac{Z_1-Z_3}{Z_1+Z_3}$

49. If two impedances, Z_1 and Z_2, are connected in parallel in an electronic circuit, the total impedance, T, is related by the statement $T=\dfrac{Z_1+Z_2}{Z_1Z_2}$. If $Z_1=10-3i$ and $Z_2=20+i$, find T.

50. The impedance in an electrical circuit is the measure of the total opposition to the flow of an electric current. The impedance, Z, in a series circuit is

$$Z=R+i(X_L-X_C),$$

where R is resistance and X_L (read "X sub L") is inductive reactance and X_C ("X sub C") is capacitive reactance. Find Z (in ohms) if $R=25$ ohms, $X_L=15+2i$ ohms, and $X_C=20+4i$ ohms.

51. (Refer to problem 50.) Suppose that $Z=12-3i$ ohms, $R=6$ ohms, and $X_L=3-8i$ ohms. Find X_C.

52. For what values of x is $\sqrt{4-x}\in C$?

53. For what values of x is $\sqrt{x-16}\in C$?

54. Show that the product of a complex number and its conjugate is always a real number by forming the product $(a+bi)(a-bi)$ and examining the result.

55. If we add two complex numbers $a+bi$ and $c+di$, we obtain a third complex number, $e+fi$. For addition, $e=a+c$ and $f=b+d$. We can see this in the following derivation:

$$\begin{aligned}(a+bi)+(c+di)&=(a+c)+(bi+di)\\&=(a+c)+(b+d)i.\end{aligned}$$

In other words,

$$(a+bi)+(c+di)=(a+c)+(b+d)i$$

is the rule for the addition of two complex numbers. Derive the rules for the subtraction, multiplication, and division of two complex numbers.

56. Julia sets are sets of numbers that appear in the study of fractals, an area of mathematics that has become much more interesting with the advent of modern computer graphics. Fractals are simple formulas that can actually generate complex graphics images. Julia sets are generated by calculating the value of the expression $z^2 + c$ repeatedly, for some fixed value of c and some starting value of z. The value calculated is the value used in the next computation of $z^2 + c$. For example, if $c = 1 + i$ and the first value of z (called the "seed value") is $2 - i$, then we proceed as shown.

z	$z^2 + c$	New value of z
$2 - i$	$(2 - i)^2 + (1 + i)$	$4 - 3i$
$4 - 3i$	$(4 - 3i)^2 + (1 + i)$	$-24 - 23i$
$-24 - 23i$	$(-24 - 23i)^2 + (1 + i)$	$48 - 1{,}103i$
	etc.	

This process produces the sequence of values $2 - i$, $4 - 3i$, $-24 - 23i$, ...

Calculate the first four values in the Julia set formed with $c = 2 - i$ and a seed value of $1 + 2i$. (The seed is the first value, so calculate three more values.)

57. See problem 56. Let an initial value of z be $0.5 - 0.2i$, and let c be $0.1 + 0.05i$. Write a program for a programmable calculator or computer to compute the successive values in the Julia set created by these values. Use the program to calculate the first 20 values. Look for a pattern.

Skill and review

1. Simplify $-5[3x - 2(1 - 4x)]$.
2. For what value(s) of x is the statement $x + 5 = 12$ true?
3. For what value(s) of x is the statement $5x = 20$ true?
4. For what value(s) of x is the statement $\frac{x}{6} = 48$ true?
5. Is the statement $3(2 - 3x) = 1 - 10x$ true when x represents -5?
6. For what value(s) of x is the statement $x + x = 2x$ true?
7. The formula $C = \frac{5}{9}(F - 32)$ converts a temperature in degrees Fahrenheit (F) to one in degrees centigrade (C). Convert 72° F to centigrade.
8. Multiply $0.06(1{,}000 - 2x)$.
9. 8% = **a.** 800 **b.** 80 **c.** 0.8 **d.** 0.08
10. Find 8% of 12,000.
11. Add 6% of 4,000 to 10% of 12,000.

Chapter 1 summary

Definitions

- Some important sets
  - N **natural numbers** $= \{1, 2, 3, \ldots\}$
  - W **whole numbers** $= \{0, 1, 2, 3, \ldots\}$
  - J **integers** $= \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
  - Q **rational numbers** $= \left\{\frac{p}{q} \,\middle|\, p \in J, q \in N\right\}$

    The decimal form of any rational number either terminates or repeats.
  - H **irrational numbers** $= \{x \mid$ The decimal representation of x does not terminate or repeat$\}$
  - R **real numbers** $= \{x \mid x \in Q$ or $x \in H\}$
  - C **complex numbers** $= \{a + bi \mid a, b \in R$, and i is the imaginary unit$\}$
- **Order of operations** Exponents, operations within symbols of grouping, multiplications and divisions from left to right, additions and subtractions from left to right.
- **Operations for fractions** (assume $a,b,c,d \in R$, and $c,d \neq 0$)

$$\frac{a}{c} \pm \frac{b}{c} = \frac{a \pm b}{c} \qquad \frac{a}{c} \pm \frac{b}{d} = \frac{ad \pm bc}{cd}$$

$$\frac{a}{c} \cdot \frac{b}{d} = \frac{ab}{cd} \qquad \frac{a}{c} \div \frac{b}{d} = \frac{a}{c} \cdot \frac{d}{b},\ b \neq 0$$

- **Absolute value** $|x| = \begin{cases} x \text{ if } x \text{ is positive or zero} \\ -x \text{ if } x \text{ is negative} \end{cases}$
- **Zero exponent** $a^0 = 1$ if $a \neq 0$
- **Negative exponent** If $n \in R$, then $a^{-n} = \frac{1}{a^n}$ if $a \neq 0$

- **Rational exponent** $a^{\frac{1}{n}} = \sqrt[n]{a}$ if $a \in R$, $n \in N$, and $\sqrt[n]{a} \in R$
  $a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$ if $m,n \in N$ and $\sqrt[n]{a} \in R$
- **Prime number** Any natural number greater than one that is divisible only by itself and one. The first primes are
  2, 3, 5, 7, 11, 13, 17, 19, 23, 29, ...
- $\boldsymbol{i = \sqrt{-1}}$; i is the imaginary unit.
- $\sqrt{-b} = i\sqrt{b}$ if $b \in R$, $b > 0$
- $\boldsymbol{a - bi}$ is the **complex conjugate** of $a + bi$.

Rules

- **Absolute value**
  [1] $|a| \geq 0$ [2] $|-a| = |a|$
  [3] $|a| \cdot |b| = |ab|$ [4] $\dfrac{|a|}{|b|} = \left|\dfrac{a}{b}\right|$
  [5] $|a - b| = |b - a|$
- **Real exponents** If $a, b, m, n \in R$ and no variable represents zero where division by that variable is indicated, then
  [1] $a^m a^n = a^{m+n}$ [2] $\dfrac{a^m}{a^n} = a^{m-n}$ [3] $(ab)^m = a^m b^m$
  [4] $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$ [5] $(a^m)^n = a^{mn}$
- **Factoring** The general types of factoring are
  Greatest common factor
  Difference of two squares
  $$a^2 - b^2 = (a - b)(a + b)$$
  Grouping
  Quadratic trinomial
  The difference and sum of two cubes
  $$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$
  $$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$
- **Fundamental principle of rational expressions** If P, Q, and R are polynomials, $Q \neq 0$, and $R \neq 0$, then $\dfrac{P}{Q} = \dfrac{P \cdot R}{Q \cdot R}$.
- **Multiplication of rational expressions** If P, Q, R, and S are polynomials, $Q \neq 0$, and $S \neq 0$, then $\dfrac{P}{Q} \cdot \dfrac{R}{S} = \dfrac{PR}{QS}$.
- **Division of rational expressions** If P, Q, R, and S are polynomials, $Q \neq 0$, $R \neq 0$, and $S \neq 0$, then $\dfrac{P}{Q} \div \dfrac{R}{S} = \dfrac{P}{Q} \cdot \dfrac{S}{R}$.
- **Addition/subtraction of rational expressions** If P, Q, R, and S are polynomials, $Q \neq 0$, and $S \neq 0$, then
  [1] $\dfrac{P}{Q} \pm \dfrac{R}{Q} = \dfrac{P \pm R}{Q}$ [2] $\dfrac{P}{Q} \pm \dfrac{R}{S} = \dfrac{PS \pm QR}{QS}$
- ***n*th root of *n*th power** $\sqrt[n]{a^n} = \begin{cases} |a| & \text{if } n \text{ is even} \\ a & \text{if } n \text{ is odd} \end{cases}$
- **Square root of a square** If $x \in R$ then $\sqrt{x^2} = |x|$.
- **Product property of radicals** If $a \geq 0$, $b \geq 0$, and $n \in N$, then $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$.
- **Quotient property of radicals** If $a \geq 0$, $b > 0$, and $n \in N$, then $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$.
- **Index/exponent common factor property** Given $\sqrt[n]{a^m}$ and $a \geq 0$. If m and n have a common factor, this factor can be divided from m and from n.
- $i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$; the value of i^n, n a whole number, can only have one of these four values.

Chapter 1 review

[1–1] The following sets are given in set-builder notation; write the sets as a list of the elements.

1. $\{x \mid x > -3 \text{ and } x < 8 \text{ and } x \in W\}$
2. $\{x \mid x \in W \text{ and } x \in N\}$
3. $\left\{\dfrac{x}{x+1} \,\middle|\, x \in \{1, 2, 3, \ldots, 100\}\right\}$

Give the decimal form of each rational number.

4. $\frac{5}{12}$
5. $\frac{3}{13}$

Simplify the given algebraic expressions.

6. $-\dfrac{5(2-8)^2}{6} - \dfrac{14(3-14)}{8}$
7. $\frac{5}{6} - \frac{3}{8} - \frac{3}{4}$
8. $\dfrac{3-8}{15} \div \dfrac{5}{(4+3^2) - \frac{10}{3}}$
9. $\dfrac{5(5 - 2(9-7) + \frac{4}{5}) - (8-12)(2-8)}{5(3-7)^3 - (3-7)^2}$
10. $\dfrac{3x}{5a} - \dfrac{y}{4b}$
11. $\dfrac{a+3}{4a} + \dfrac{3b+2}{b}$
12. $\dfrac{3x}{2y} \div \dfrac{5y}{2x} \cdot \dfrac{x}{5y}$
13. $\left(\dfrac{2a}{b} + \dfrac{b}{3a}\right) \cdot \left(\dfrac{3b}{5a}\right)$

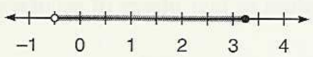
In problems 14 and 15 an interval is indicated in set-builder notation; give the interval notation and the graph of the set.

14. $\{z \mid -2\frac{1}{2} \leq z < -\frac{3}{4}\}$ **15.** $\{y \mid -\frac{1}{4} < y\}$

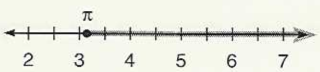
In problems 16 and 17 an interval is indicated in interval notation; give the set-builder notation and the graph of the set.

16. $(-\infty, -1]$ **17.** $\left[-\frac{\pi}{3}, \pi\right)$

18. Describe the interval in the figure in both set-builder and interval notation.



19. Describe the interval in the figure in both set-builder and interval notation.



In problems 20–22 express the value of the expression without the absolute value symbol.

20. $-\left|-\frac{1}{2}\right|$ **21.** $|-\pi - 9|$ **22.** $-|\sqrt{2} - 5|$

In problems 23–26 use the rules for absolute value to rewrite the expression with as few factors as possible in the absolute value operator.

23. $|-5x^2|$ **24.** $\left|\frac{3x^2}{2y^3}\right|$ **25.** $|10x - 5|$

26. $-|(x-2)^2(x+1)|$

[1–2] Use the rules for exponents to simplify the following expressions.

27. $(3x^2)^3y^{-3}$ **28.** $-3xy^{-2}$ **29.** $5^{-2}x^3y^2$

30. $(-3x^5y)(-2x^{-4}y^{-2})$ **31.** $(-3^2xy^{-5})(2^2x^5y^{-1}z^0)$

32. $\frac{-3^2x^{-4}y}{-3xy^{-4}}$ **33.** $\left(\frac{-2x^{-2}y^{-1}}{8x^{-2}y^{-3}}\right)^3$ **34.** $\left(\frac{2a^{-2}}{12a^{-12}}\right)^{-2}$

35. $x^{3-4n}x^{2n-3}$ **36.** $\left(\frac{x^{3n}y^{3-n}}{x^{-n}y^{n-3}}\right)^2$

Convert each number into scientific notation.

37. 42,182,000,000,000,000

38. −0.000 000 000 046 05

Convert each number given in scientific notation into a decimal number.

39. 4.052×10^{-7} **40.** -3.409×10^{11}

In problems 41–43 find the value that each expression represents, assuming that $a = 2$ and $b = -6$.

41. $a^3 - 2a^2 + 12a + 3$ **42.** $(6a^2 - 2a + 1)(a - 1)^2$

43. $-2b(\frac{1}{3}(4(\frac{1}{2}b(-b + 4) - 2) - 7) + 2) + 1$

Multiply.

44. $-3x^3(5x^3 + 7x - \frac{1}{3} - 2x^{-3})$

45. $(a - 5)^2(5a + 1)$

46. $(x^2 - 5x + 5)(-3x^3 - 2x^2 + 3)$

Perform the indicated divisions.

47. $\frac{-12x^6 - 4x^4 + 4x^2}{4x^2}$

48. $\frac{30a^8b^4 + 9a^4b^4 + 12a^4b^{12}}{-3a^4b^4}$

49. $\frac{y^4 - 1}{y - 1}$

50. $\frac{4x^4 - 4x^3 - 5x^2 - 10x - 5}{2x + 1}$

[1–3] Factor the following expressions.

51. $25x^3 - x^5$ **52.** $x^2 + 13x + 36$

53. $8a^3 - 14a^2 + 5a$ **54.** $3a^2b^2 + 2ab - 8$

55. $8a^3b + 125b^4$

56. $45ax^2(x^2 - 1) - 5a(x^2 - 1)$

57. $5x^2 - 51xy + 10y^2$ **58.** $(a - b)^2 - (2x + y)^2$

59. $54x^6 - 2y^3$ **60.** $12x^2 - 48y^2$

61. $-2by + 3ax - bx + 6ay$ **62.** $8a^9 - b^3c^3$

63. $7a^2 - 32a - 21$ **64.** $4x^4 - 24a^2x^2 + 36a^4$

65. $3(x^3 - 1)^2 - 2(x^3 - 1) - 8$

66. $4b(x + 3y) - 16a^2b(x + 3y)$

67. $a^2 - 4(x + 5y)^2$ **68.** $3x^2 - 8x - 91$

69. $3x^5y^9 + 81x^2z^6$

70. $x^2(x^2 - 1) - 4x(x^2 - 1) - 4(1 - x^2)$

[1–4] Reduce each rational expression to lowest terms.

71. $\frac{3a^2 - a}{9a^2 - 1}$ **72.** $\frac{2y - 3xy}{6x^2 + 5x - 6}$

73. $\frac{4x^2 - 1}{8x^3 - 1}$ **74.** $\frac{2x(x - 2)^2 + (x - 2)^2}{4x^4 - 17x^2 + 4}$

Perform the indicated operations.

75. $\frac{x - 2y}{5x} + \frac{3x - y}{3y}$ **76.** $\frac{x - 5}{x - 2} + \frac{6 - 2x}{4 - 2x}$

77. $\frac{1}{4x} - \frac{2x}{4x - 5}$ **78.** $\frac{18b^2}{14a^3} \cdot \frac{28a^3}{3b^3}$

79. $\frac{2x^3 - 4x^2 + 2x}{x^2 + 3x - 4} \cdot \frac{x^2 - 16}{4x^2 - x - 3}$

80. $\frac{x^2 - x - 6}{x^2 - 2x - 3} \div (x^2 + 4x + 4)$

81. $\frac{3x}{x^2 - 5x + 6} - \frac{2x - 5}{x - 3}$

82. $\frac{2x - 1}{x^2 + 2x - 3} + \frac{x + 1}{9 - x^2} - \frac{3}{x - 3}$

Simplify each complex rational expression.

83. $\dfrac{\dfrac{5}{2x} - \dfrac{3}{y}}{\dfrac{1}{x} + \dfrac{2}{y}}$ **84.** $\dfrac{\dfrac{3a - 2b}{2a} + \dfrac{5}{3}}{\dfrac{2a + 3b}{4a} - \dfrac{5}{6a}}$ **85.** $\dfrac{\dfrac{3}{a - b} - 2}{5 - \dfrac{3}{a - b}}$

[1–5] Find the indicated root.

86. $\sqrt[5]{-32}$ **87.** $\sqrt[4]{256}$

Simplify the following radical expressions. Assume that all variables represent nonnegative real numbers.

88. $\sqrt[3]{432}$ **89.** $\sqrt{54x^4y^7z^2}$ **90.** $\sqrt[3]{48a^6b^5c^8}$

91. $(\sqrt[3]{9a^4b^2})^2$ **92.** $\sqrt[4]{25a^2b^4c}\sqrt[4]{25a^2b^5c^2}$

93. $\sqrt[6]{a^4b^6}$ **94.** $\sqrt{128a^{15}b^{10}}$

95. $\dfrac{18a}{\sqrt{108a^3}}$ **96.** $\sqrt[3]{\dfrac{16x^4y^7}{25w^5z^2}}$

Simplify the following radical expressions. Assume that all variables represent nonnegative real numbers.

97. $-\sqrt{72a^2b^3} + 3\sqrt{50a^2b^3}$

98. $\sqrt[4]{48b^3c^6} - \sqrt[4]{243b^3c^6}$

99. $\sqrt{3xy^3}(\sqrt{3xy} - \sqrt{3x} + \sqrt{6y})$

100. $(\sqrt[4]{9x^2} - 3\sqrt[4]{3x^3})^2$

Rationalize the denominators. Assume that all variables represent nonnegative real numbers.

101. $\dfrac{\sqrt{6x} - 5}{\sqrt{6x} + \sqrt{2}}$ **102.** $\dfrac{a + \sqrt{a}}{\sqrt{a^3} + \sqrt{ab}}$

Perform the indicated operations and simplify.

103. $\dfrac{1}{4}\left(\dfrac{3}{2\sqrt{2}}\right) - \dfrac{\sqrt{3}}{4}\left(-\dfrac{1}{\sqrt{2}}\right)$ **104.** $\sqrt{\dfrac{3 - \dfrac{\sqrt{2}}{3}}{3}}$

[1–6] Rewrite each expression in terms of radicals. Simplify if possible. Assume all variables represent nonnegative values.

105. $(25x^3)^{\frac{1}{2}}$ **106.** $(16x^8)^{\frac{3}{4}}$ **107.** $(81x^6)^{-\frac{1}{4}}$

Simplify. Variables may represent negative values.

108. $\sqrt{8x^6y^9}$ **109.** $\sqrt{\dfrac{x^6}{8y^{10}}}$

110. $\sqrt{16x^6(x - 3)^2}$

Simplify. Assume all variables represent nonnegative values.

111. $\left(x^{\frac{3}{4}}y^{-\frac{1}{4}}\right)\left(x^{\frac{2}{3}}y^{\frac{1}{2}}\right)$ **112.** $\left(8x^{\frac{4}{5}}y^{-\frac{3}{4}}z\right)^{\frac{2}{3}}$

113. $\dfrac{\left(4x^{-\frac{3}{2}}y\right)^{-\frac{1}{2}}}{x^{\frac{1}{2}}y^{\frac{3}{4}}}$ **114.** $\left(\dfrac{64x^{-\frac{3}{5}}y^{-\frac{3}{4}}}{8z^{\frac{3}{4}}}\right)^{-\frac{8}{3}}$

115. $\left(x^{\frac{3a}{b}}y^{\frac{2a}{c}}\right)^{\frac{bc}{6a}}$ **116.** $\left(\dfrac{8x^{\frac{m+n}{m}}}{2x^{\frac{n}{3m}}y^{-\frac{n}{3m}}}\right)^{3m}$

Compute the following values to four places of accuracy.

117. $(-356)^{\frac{2}{3}}$ **118.** $\sqrt[4]{25^8}$

119. $\sqrt[5]{8.2^3}$ **120.** $(\sqrt[4]{200})^{\frac{1}{3}}$

[1–7] Perform the operations shown on the complex numbers.

121. $(-13 + 12i) - (6\frac{1}{2} - 3i)$

122. $(-8 + 3i)(2 - 7i)$

123. $(\frac{2}{3} - 3i)(-3 + \frac{1}{3}i) - (\frac{1}{3} - 3i)^2$

124. $\dfrac{3 - 6i}{2 + 5i}$ **125.** $\dfrac{-3 + 2i}{i(4 - i)}$

Simplify the following expressions.

126. $(1 - \sqrt{-8})(4 + \sqrt{-8})$

127. $\dfrac{6 + 3\sqrt{-12}}{4 - \sqrt{-12}}$ **128.** $(2 - \sqrt{-2})^4$

In problems 129 and 130 compute the value of the expression if $A = 1 - 5i$ and $B = -2 + 2i$.

129. $\dfrac{B - A}{B^2 - A^2}$ **130.** $A(B + A)$

131. If two impedances, A and B, are connected in parallel in an electronic circuit the total impedance, T, is related by the statement $\dfrac{1}{T} = \dfrac{1}{A} + \dfrac{1}{B}$. If $A = 1 - 3i$ and $B = 4 + i$, find T.

132. Evaluate

$$x^8 - 5x^7 + x^6 - 3x^5 + 2x^4 + x^3 + 5x^2 - 11x + 1$$

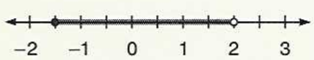
for $x = i$.

Chapter 1 test

1. Write the set as a list of elements $\left\{\frac{x}{x+2} \,\middle|\, x \in \{1, 2, 3, 4, 5\}\right\}$.

Simplify the given algebraic expressions.

2. $-\frac{2(2-5)^3}{6} - (2^2 - 3^2)$
3. $\frac{5}{4} - \frac{3}{8} - \frac{2}{3}$
4. $\frac{a+3b}{4b} - \frac{3b+2a}{a}$
5. $\left(\frac{2a}{b} + \frac{b}{3a}\right) \cdot \left(\frac{3b}{5a}\right)$
6. Give the interval notation and the graph of the set $\left\{z \,\middle|\, -2\frac{1}{2} < z \le -\frac{3}{4}\right\}$.
7. Give the set-builder notation and the graph of the set $[-3, \infty)$.
8. Describe the interval in both set-builder and interval notation.



Express the value of the expression without the absolute value symbol.

9. $-|-4|$
10. $|-\pi + 2|$
11. Rewrite the expression with as few factors as possible in the absolute value operator. $|-3x^2y|$

Simplify the following expressions.

12. $(-2x^5y)(-3x^4y^2)$
13. $(-2^2x^{-1}y^5)(2^2x^5y^{-1}z^0)$
14. $\frac{3^2x^4y}{-3xy^{-4}}$
15. $\left(\frac{2a^{-2}}{12a^{-12}}\right)^{-2}$
16. $x^{-3-2n}x^{n+3}$
17. Convert into scientific notation: 205,000,000,000.
18. Convert into a decimal number: 2.13×10^{-4}.

In problems 19 and 20 find the value that each expression represents, assuming that $a = -\frac{2}{3}$, $b = 2$.

19. $(9a^2 - 6a + 1)(a + 1)$
20. $\frac{1}{3}(4(\frac{1}{2}b(-b + 4) - 2) - 7)$

Multiply.

21. $-2x^3(x^3 + \frac{1}{2}x - 3 - 2x^{-3})$
22. $(a - 5)^2(a + 5)^2$
23. Divide: $\frac{2x^3 - x^2 + 4x - 5}{x - 2}$.

Factor the following expressions.

24. $4a^3 - 16a$
25. $9x^2 - 3x - 2$
26. $x^4 - 16$
27. $64x^6 - 1$
28. $(x - 2)^2 + 2(x - 2) - 3$
29. $3ac - 2bd + ad - 6bc$

Reduce each rational expression to lowest terms.

30. $\frac{2x^2 - 2x}{2x^2 - 2}$
31. $\frac{2x^2 + x - 1}{4x^2 - 1}$

Perform the indicated operations.

32. $\frac{x-5}{x-2} - \frac{6+3x}{6-3x}$
33. $\frac{x}{2x+1} - \frac{x-1}{3x}$
34. $\frac{x^2 - 2x + 1}{x^2 - 4} \cdot \frac{x^2 - 4x + 4}{x^2 - 1}$

Simplify each complex rational expression.

35. $\dfrac{\frac{2}{3a} - \frac{3}{b}}{\frac{2}{3ab} + 2}$
36. $\dfrac{3 - \frac{3}{a-b}}{3 + \frac{3}{a-b}}$

Simplify the following radical expressions. Assume that all variables represent nonnegative real numbers.

37. $\sqrt[3]{128}$
38. $\sqrt{50x^4y^3z}$
39. $(\sqrt[3]{16ab^5})^2$
40. $\sqrt[4]{36ab^4c^2}\sqrt[4]{36a^2b^5c^2}$
41. $\frac{12x^3}{\sqrt{24x^5}}$
42. $\sqrt[3]{\frac{8x^4}{y^2z}}$

Simplify the following radical expressions. Assume that all variables represent nonnegative real numbers.

43. $\sqrt{45a^3b} - a\sqrt{20ab} + \sqrt{5a^3b}$
44. $-\sqrt{2x^2y}(\sqrt{2xy} - \sqrt{8x} + \sqrt{6y})$
45. Rationalize the denominator of $\frac{3\sqrt{a} - 3}{\sqrt{6a} + \sqrt{3}}$. Assume that all variables represent nonnegative real numbers.
46. Perform the operations and simplify: $$\frac{1}{\sqrt{3}} \cdot \frac{1}{2} - \frac{\sqrt{5}}{\sqrt{3}}\left(-\frac{\sqrt{3}}{2}\right).$$

Rewrite each expression in terms of radicals. Simplify if possible. Assume all variables represent nonnegative values.

47. $(16x^3)^{\frac{1}{3}}$
48. $(16x^{11}y^4)^{-\frac{1}{4}}$

Simplify. Variables may represent negative values.

49. $\sqrt{20x^6y^8}$ **50.** $\sqrt{25x^4(x-3)^2}$

Simplify. Assume all variables represent nonnegative values.

51. $\left(x^{\frac{1}{4}}y^{-\frac{3}{4}}\right)\left(x^{\frac{3}{4}}y^{\frac{1}{2}}\right)$ **52.** $\left(27x^{\frac{3}{4}}y^{-\frac{3}{8}}z\right)^{\frac{4}{3}}$

53. $\dfrac{\left(8x^{\frac{3}{2}}y\right)^{-\frac{1}{3}}}{2x^{-\frac{1}{3}}y^{\frac{2}{3}}}$ **54.** $\left(\dfrac{64x^{-\frac{3}{2}}y^{-\frac{3}{4}}}{8z^{\frac{3}{4}}}\right)^{\frac{8}{3}}$

55. $\left(a^{\frac{m}{n}}b^{\frac{2}{n}}\right)^{\frac{n}{2m}}$

56. Compute $91^{-\frac{2}{3}}$ to four places of accuracy.

Perform the operations shown on the complex numbers.

57. $(-8+3i)^2(2-7i)$ **58.** $\dfrac{3-6i}{2+5i}$

Simplify the following expressions.

59. $(\sqrt{-4}-\sqrt{-9})(3+\sqrt{-12})$

60. $\dfrac{6+\sqrt{-12}}{3+\sqrt{-12}}$

61. If two impedances, Z_1 and Z_2, are connected in parallel in an electronic circuit, the total impedance, T, is related by the statement $\dfrac{1}{T}=\dfrac{1}{Z_1}+\dfrac{1}{Z_2}$. If $Z_1 = 2+4i$ and $Z_2 = 1+2i$, find T.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